THERMAL STRESSES IN THIN SHALLOW SHELLS

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CERTIFICATE

Certified that this work on "Thermal stresses in thin shallow shells" by Ram Sajiwan, as a partial fulfilment towards the degree of Master of Technology in Civil Engineering, has been carried out under my supervision and that this has not been submitted else where for a degree.

(Y.C.Das)
Professor
and chairman of Thesis Committee

Kanpur:
October, 1969

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SYNOPSIS

The present work involves the study of thermal stresses in thin shallow shells. Chapter I gives the introduction about the thermal stresses and amount and type of work done in thin shallow shells. Chapter II deals with the variation of total potential energy in deriving the basic equations for general thin shells and possible boundary conditions in thin shells. Chapter III deals with the making of nonhomogeneous boundary conditions to homogeneous conditions by Mindlin and Goodman technique. This chapter gives the solutions for two parallel edges simply supported and two other edges supported in any manner. Solutions have been given for (a) all four edges simply supported and (b) two parallel edges simply supported and remaining two edges clamped, where constants have been obtained with the help of Huang's Technique. An example illustrates the method.

A flow chart and a computer programme for (a) all edges simply supported and (b) two parallel edges simply supported and remaining two edges clamped, are given in Appendix (B) and (C) respectively. Tables and graphs are presented for deflections and moments in terms of dimensionless parameters for a/b ratio from 1.0 to 2.0 at interval of 0.2 and for a^2/hR_1 ratio from 5.0 to 20.0 at interval of 5.0 for elliptic paraboloids.

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NOTATIONS

A_1, A_2	Lame's constants
C_E	Specific heat at constant deformation
D	d/dy (operator)
D_1	$\frac{Eh^3}{12(1-\nu^2)}$ flexural rigidity of shell
E	Modulus of elasticity in tension or compression
ν	Poisson's ratio
k	Thermal conductivity of the solid
M_1, M_2, M_{12}	Bending and twisting moments per unit distance in shell.
M_T	$\alpha E \int_{-h/2}^{h/2} T.z.\ dz$
N_T	$\alpha E \int_{-h/2}^{h/2} T.\ dz$
q_1, q_2, q_n	Components of intensity of surface loading
T	Temperature rise.
T_1, T_2, S	Normal and shearing forces per unit distance in middle surface of shell.
u. v. w.	Components of displacements at distance z from middle plane.
α	Coefficient of linear expansion
β_1, β_2, z	Curvilinear Coordinates
$\sigma_1, \sigma_2, \sigma_3$	Normal components of stress parallel to β_1, β_2 and z-axis
$\tau_{12}, \tau_{13}, \tau_{23}$	Shearing stress components in curvilinear coordinates.

$\epsilon_1, \epsilon_2, \epsilon_3$	Unit elongations in β_1, β_2, and z-directions.
$\epsilon_1^o, \epsilon_2^o$	Unit elongations in β_1, β_2 directions on the middle surface.
$\gamma_{12}, \gamma_{23}, \gamma_{13}$	Shearing strain components in curvilinear coordinates.
ϕ	Stress function
ρ	Mass density
λ, μ	Lame's constants
$\nabla^2(\text{---})$	$\frac{1}{A_1A_2}\left\{\frac{\partial}{\partial\beta_1}\left[\frac{A_2}{A_1}\frac{\partial(\text{---})}{\partial\beta_1}\right]+\frac{\partial}{\partial\beta_2}\left[\frac{A_1}{A_2}\frac{\partial(\text{---})}{\partial\beta_2}\right]\right\}$
$L(\text{---})$	$\frac{1}{A_1A_2}\left\{\frac{\partial}{\partial\beta_1}\left[\frac{1}{R_2}\frac{A_2}{A_1}\frac{\partial(\text{---})}{\partial\beta_1}\right]+\frac{\partial}{\partial\beta_2}\left[\frac{1}{R_1}\frac{A_1}{A_2}\frac{\partial(\text{---})}{\partial\beta_2}\right]\right\}$

CHAPTER I

INTRODUCTION

In modern design, there is a lot of attention towards the thermal stresses chiefly because of many engineering components which fail because of it. Thermal stresses play an important role in the design of gas or steam turbines, super sonic aircrafts, nuclear reactors, rocket engines, missiles, airframe structures, high-temperature oilrefining plants and structures operating at elevated temperatures.

Thermal stresses are arising from temperature effects. When temperature gradient is applied to material of the body, the various fibers tend to expand different amounts. To enable the body to remain continuous, a system of thermal strains and associated thermal stresses may be introduced which depend on the shape of the body and the temperature distribution inside the body. Thermal stresses are also introduced in cases where no temperature gradient exists if one part consists of several materials having different coefficient of expansions, or if free contraction or expansion is prevented by external constraints. As an example, a free plate having a temperature distribution varying with depth only will bend into a curved plate. Thermal stresses will be introduced in a free plate made of two materials, having different coefficient of expansions, with uniform temperature distribution. And thermal stresses are

also set in constrained edges plate having uniform temperature distribution.

The basic energy equation (1)** derived from 'Law of conservation of energy' under the action of heat is given as

$$K\, T_{,ii} = \rho C_E\, \dot{T} + (3\lambda + 2\mu)\, \alpha T_0\, \dot{\epsilon}_{kk} \quad \ldots\ldots(1)$$

This is a coupled equation, which required simultaneous determination of temperature distribution and deformations.

This boundary value problem has considerable mathematical defficulty, since it combines the theories of heat conduction and elasticity under transient conditions. But fortunately in most of engineering problems, it is possible to introduce certain simplifying assumptions without significant error. The principal simplifying assumption is omission of mechanical coupling term in energy equation and giving rise some form of Fourier heat conduction equation, which is independent of stresses and deformations. If variations of strain in a body are produced by some external agency, these variations of strain are accompanied by variations in temperature and consequently flow of heat. This gives rise to an increase of entropy and corresponding increase of energy stored in a mechanically irrecoverable manner and this is thermoelastic dissipation, which requires for its study the use of coupled heat equation. The deformations due to external loads are accompanied by only small changes in temperature. It is then reasonable to calculate these deformations without taking account of

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**Numbers refer to entries in the list of references.

thermal expansion. Similarly, if strains are produced in a body by non-uniform temperature distribution, it is clear that influence of these strains on the temperature itself is not too large. Thus omission of mechanical coupling term in energy equation would be meaning less.

Second simpligying assumption is omission of inertia term in an equation of motion. Duhamel had shown that the time rate of change of temperature is slow enough so that this term should not be significant. Danilovskaya [2] and Boley have also showed that omission of inertia term is meaningless The two simplifying assumptions give rise to uncoupled quasi-static theory. This uncoupled quasi-static theory is divided into two distinct subjects, and known as theories of heat conduction and thermoelasticity. Solution of heat conduction equation gives the distribution of temperature inside the body.

In the present case, it is assumed that temperature distribution in side the thin shallow shell is known before hand. It is taken that temperature is anti-symmetric about the middle plane of the thin shallow shell but otherwise arbitrary.

Due to elastic constants such as Young's Modulus,

The thermal stresses are related in a complicated manner to the temperature distribution. Mechanical and Metallurgical properties may be changed with the temperature. It is assumed that elastic constants do not depend upon the temperature and material behaves elastically at all times.

Only Gradowczyk[3] has tried to derive the governing equations for thin shallow shells. He has taken Navier type solution for deflection of thin shallow shell. Navier type solution is possible only for all edges simply supported. But Navier type solution is in termsof double trigonometric series which behaves poorly near the end point of interval. Gradowczyk has not solved completly for any type of boundary conditions.

Stress function and deflection of thin shallow shell subjected to temperature must satisfy the following two equations.

$$D_1 \; \nabla^2 \; \nabla^2 \; W - L\,(\,\theta\,) = q_n - \nabla^2 \frac{M_T}{(1-\nu\,)} \quad \ldots.(2)$$

and

$$\nabla^2 \nabla^2 \theta + E\,h\,L\,(W) = \nabla^2 N_T \qquad \ldots.(3)$$

Stress and deflection functions are separated into different equations. Solutions of these equations

are obtained as Levy's type solutions which are in terms of hyperbolic and trigonometric functions. It gives better convergence near the end in comparison to Navier type solutions which is in terms of double trigonometric series.

The boundary conditions considered are:

a. Two opposite edges of thin shallow shell simply supported and remaining two edges supported in any manner.

b. All four edges of thin shallow shell simply supported.

c. Two opposite edges of thin shallow shell simply supported and remaining two edges clamped.

CHAPTER II

FORMULATION OF BASIC EQUATIONS

Following assumptions have been made in the derivation of basic equations:

1. The material is homogeneous, isotropic and obeys Hook's Laws.

2. Planes perpendicular to middle surface before bending, remain plane and perpendicular to middle surface after bending and do not stretch in their length.

i.e. $\gamma_{13} = 0$, $\gamma_{23} = 0$, $\epsilon_3 = 0$

3. Normal stresses on planes parallel to middle surface are small in comparision to other stresses and these normal stresses are neglected.

i.e. $\sigma_3 = 0$

4. Deflections are small compared to thickness and the effect of loads in the plane of shell is neglected and coordinates of a particle before and after deformation are the same.

5. The elastic constants E, ν and the coefficient of thermal expansion are independent of temperature.

6. The thickness of the thin shell is very small in comparision with the least radius of curvature of the middle surface.

i.e. $1 \pm z/R_1 = 1$ and $1 \pm z/R_2 = 1$

7. Thickness of the shell is uniform.

Let β_1 and β_2 be the coordinate of point on the middle surface of thin shell and z be the distance measured along the outward normal to the middle surface. The element of the thin shell acted by different forces is shown in fig.(1). Introduce approximations for the displacements (4) as follows.

$$u = u_0 (\beta_1, \beta_2) + z \phi \qquad \text{..................(4a)}$$

$$v = v_0 (\beta_1, \beta_2) + z \psi \qquad \text{..................(4b)}$$

$$w = w (\beta_1, \beta_2) \qquad \text{..................(4c)}$$

where u_0 and v_0 are the components of displacements at the middle surface; ϕ and ψ are the change of slopes of the normals to the middle surface.

Define following stress resultants using Love's first approximation.

$$T_1 = \int_{-h/2}^{h/2} \sigma_1 \, dz \qquad M_1 = \int_{-h/2}^{h/2} \sigma_1 z \, dz$$

$$T_2 = \int_{-h/2}^{h/2} \sigma_2 \, dz \qquad M_2 = \int_{-h/2}^{h/2} \sigma_2 z \, dz$$

$$S = \int_{-h/2}^{h/2} \tau_{12} \, dz \qquad M_{12} = \int_{-h/2}^{h/2} \tau_{12} z \, dz \qquad \text{—— 5(a to}$$

$$N_T = E \alpha \int_{-h/2}^{h/2} T \, dz \qquad M_T = E \alpha \int_{-h/2}^{h/2} T z \, dz$$

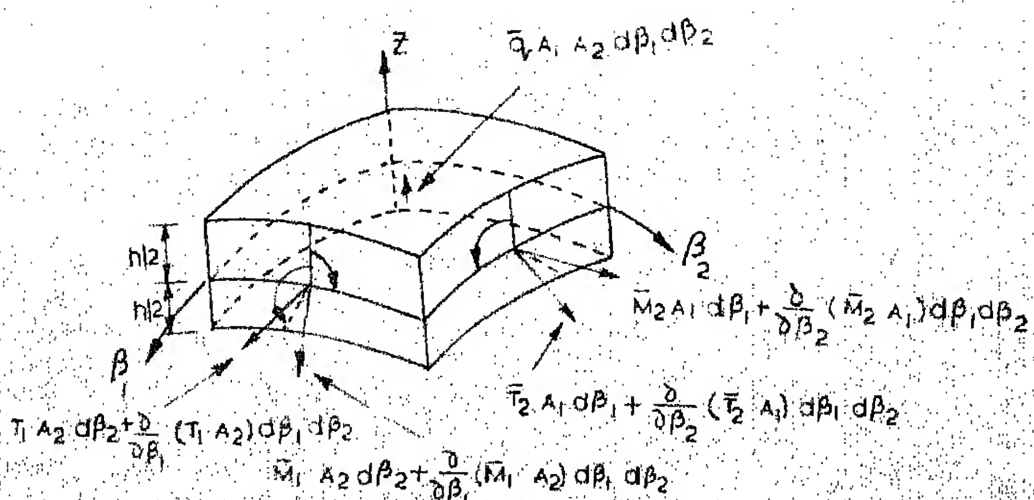

FIG. 1 ELEMENT OF SHELL AT EQUILIBRIUM

stress strain relations are modified to take into account the strains due to temperature and other basic equations of thermoelasticity are same as the ordinary thin shell theory.(5)

$$e_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) + \alpha T \qquad \text{.................(6a)}$$

$$e_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1) + \alpha T \qquad \text{.................(6b)}$$

and $$\gamma_{12} = \frac{2(1+\nu)}{E}\tau_{12} \qquad \text{.................(6c)}$$

or

$$\sigma_1 = \frac{E}{(1-\nu^2)}\cdot(e_1 + \nu e_2 - (1+\nu)\alpha T) \qquad \text{.........(6d)}$$

$$\sigma_2 = \frac{E}{(1-\nu^2)}(e_2 + \nu e_1 - (1+\nu)\alpha T) \qquad \text{.........(6e)}$$

$$\tau_{12} = \frac{E}{2(1+\nu)}\gamma_{12} \qquad \text{.........(6f)}$$

strain components at any points at a distance z from the middle surface of shell in terms of middle plane strain components and curvatures are given by

$$e_1 = e_1^0 + z K_1 \qquad \gamma_{13} = 0$$

$$e_2 = e_2^0 + z K_2 \qquad \gamma_{23} = 0 \qquad 6(g\text{-}m)$$

$$\gamma_{12} = \gamma_{12}^{0} + z\,\kappa_{12} \qquad e_3 = 0 \quad .$$

(8b)

$$\varepsilon_1^0 = \frac{v_0}{A_1 A_2}\,\frac{\partial A_1}{\partial \beta_2} + \frac{w}{R_1} + \frac{1}{A_1}\,\frac{\partial u_0}{\partial \beta_1}$$

(8c)

$$\varepsilon_2^0 = \frac{u_0}{A_1 A_2}\,\frac{\partial}{\partial \beta_1} A_2 + \frac{w}{R_2} + \frac{1}{A_2}\,\frac{\partial}{\partial \beta_2}$$

(8d)

$$\gamma_{12}^0 = \frac{A_2}{A_1}\,\frac{\partial}{\partial \beta_1}\left(\frac{v_0}{A_2}\right) + \frac{A_1}{A_2}\,\frac{\partial}{\partial \beta_2}\left(\frac{u_0}{A_1}\right)$$

(8e)

$$\kappa_1 = -\frac{1}{A_1}\,\frac{\partial}{\partial \beta_1}\left(-\frac{u_0}{R_1} + \frac{1}{A_1}\,\frac{\partial w}{\partial \beta_1}\right) - \frac{1}{A_1 A_2}\left(-\frac{v_0}{R_1}\right.$$

(8f)

(9a)

$$\kappa_2 = -\frac{1}{A_2}\,\frac{\partial}{\partial \beta_2}\left(-\frac{v_0}{R_2} + \frac{1}{A_2}\,\frac{\partial w}{\partial \beta_2}\right) - \frac{1}{A_1 A_2}\left(-\right.$$

b)

$$\kappa_{12} = -\frac{A_2}{A_1}\,\frac{\partial}{\partial \beta_1}\left(-\frac{v_0}{A_2 R_2} + \frac{1}{A_2^2}\,\frac{\partial w}{\partial \beta_2}\right) - \frac{A_1}{A_2}\,\frac{\partial}{\partial \beta_2}$$

Substituting Eq. (6) into Eq. (5), the str

9c)

and stress couples in terms of the middle

and curvature are obtained as

(9d)

$$N_1 = \frac{Eh}{1-\nu^2}\left(\varepsilon_1^0 + \nu\,\varepsilon_2^0\right) - \frac{N_T}{1-\nu} \quad ..$$

$$N_2 = \frac{Eh}{(1-\nu^2)} \left(\epsilon_2^0 + \nu \epsilon_1^0 \right) - \frac{N_T}{(1-\nu)} \qquad \text{.........(8b)}$$

$$N_{12} = \frac{Eh}{2(1+\nu)} \gamma_{12}^0 \qquad \text{.......(8c)}$$

$$M_1 = D_1 (\kappa_1 + \nu \kappa_2) - \frac{M_T}{(1-\nu)} \qquad \text{........(8d)}$$

$$M_2 = D_1 (\kappa_2 + \nu \kappa_1) - \frac{M_T}{(1-\nu)} \qquad \text{........(8e)}$$

$$M_{12} = + \frac{D_1}{2} (1-\nu) \kappa_{12} \qquad \text{........(8f)}$$

$$\phi = \frac{u_0}{R_1} - \frac{1}{A_1} \frac{\partial w}{\partial \beta_1} \qquad \text{........(9a)}$$

$$\psi = \frac{v_0}{R_2} - \frac{1}{A_2} \frac{\partial w}{\partial \beta_2} \qquad \text{.......(9b)}$$

Shear force in terms of stress couples are given by

$$Q_1 = \frac{1}{A_1 A_2} \left(2 \frac{\partial A_1 M_{12}}{\partial \beta_2} + \frac{\partial A_2 M_1}{\partial \beta_1} - \frac{\partial A_2}{\partial \beta_1} M_2 \right) - \frac{1}{A_2} \frac{\partial M_{12}}{\partial \beta_2} \qquad (9c)$$

$$Q_2 = \frac{1}{A_1 A_2} \left(2 \frac{\partial A_2 M_{12}}{\partial \beta_1} + \frac{\partial A_1 M_2}{\partial \beta_2} - \frac{\partial A_1}{\partial \beta_2} M_1 \right) - \frac{1}{A_1} \frac{\partial M_{21}}{\partial \beta_1} \qquad (9d)$$

Force equilibrium equations and equilibrium equations for deflection of an elastic thin shell have been derived by minimizing the total potential energy 'V' as follows.

$$V = \frac{1}{2} \iiint (\sigma_1 e_1 + \sigma_2 e_2 + \sigma_3 e_3 + \gamma_{12} \tau_{12} + \gamma_{23} \tau_{23}$$

$$+ \gamma_{13} \tau_{13}) A_1 A_2 \, d\phi_1 \, d\phi_2 \, dz - \iint (q_1 u_0 + q_2 v_0$$

$$+ qw) A_1 A_2 \, d\phi_1 \, d\phi_2 \qquad \text{..............(10)}$$

By putting the values of $\sigma_1, \sigma_2, \sigma_3, \tau_{12}, \gamma_{23}$ and γ_{13} in Eq. (10)

$$V = \frac{E}{2(1-\nu^2)} \iiint (e_1^2 + e_2^2 + 2\nu e_1 e_2 - 2(1+\nu) \; ez \; (e_1$$

$$+ e_2) + \frac{(1-\nu)}{2} \gamma_{13}^2) \; A_1 A_2 \, d\phi_1 \, d\phi_2 \, dz -$$

$$- \iint (q_1 u_0 + q_2 v_0 + qn \, w) \; A_1 A_2 \, d\phi_1 \, d\phi_2 \text{ .... } \quad (11)$$

By putting the values of $e_1, e_2,$ and $_{12}$ in Eq. (11)

$$V = \frac{E}{2(1-\nu^2)} \iiint \left\{ (e_1^0 + e_2^0)^2 - 2(1-\nu) \; (e_1^0 \, e_2^0 - \frac{\gamma_{12}^{0\,2}}{4}) \right.$$

$$\left. + z^2 \left\{ (k_1 + k_2)^2 - 2(1-\nu) \; (k_1 k_2 - \frac{k_{12}^2}{4}) \right\} \right.$$

$$- 2(1+\nu)\,\alpha\,T\left\{\varepsilon_1^0+\varepsilon_2^0+z(k_1+k_2)\right\}\Big\}\,A_1A_2\,d\beta_1\,d\beta_2\,dz$$

$$-\iint(q_1u_0+q_2v_0+q_n w)\,A_1A_2\,d\beta_1\,d\beta_2 \quad \ldots\ldots\ldots\ldots (12)$$

Integrating with respect to z

$$V=\frac{1}{2(1-\nu^2)}\iint\Big[Eh\left\{(\varepsilon_1^0+\varepsilon_2^0)^2-2(1-\nu)\left(\varepsilon_1^0\varepsilon_2^0-\frac{\gamma_{12}^{0^2}}{4}\right)\right\}$$

$$+\frac{Eh^3}{12}\left\{(k_1+k_2)^2-2(1-\nu)\left(k_1k_2-\frac{k_{12}^2}{4}\right)\right\}$$

$$-2(1+\nu)\left\{N_T(\varepsilon_1^0+\varepsilon_2^0)+M_T(k_1+k_2)\right\}\Big]\,A_1A_2\,d\beta_1\,d\beta_2$$

$$-\iint(q_1u_0+q_2v_0+q_n w)\,A_1A_2\,d\beta_1\,d\beta_2 \quad \ldots\ldots\ldots (13)$$

Minimizing the total potential energy 'V' by variational principle, we get

$$\delta V=\iint\Big[T_1\,\delta\varepsilon_1^0+T_2\,\delta\varepsilon_2^0+S\,\delta\gamma_{12}^0+M_1\,\delta k_1+M_2\,\delta k_2$$

$$+H_{12}\,\delta k_{12}-(q_1\,\delta u_0+q_2\,\delta v_0+q_n\,\delta w)\Big]\,A_1A_2\,d\beta_1\,d\beta_2$$

$$=0 \quad \ldots\ldots\ldots (14)$$

Changes incurvature and twist caused by displacement components tangential to the middle surface are very small and neglected in k_1, k_2 and k_{12}.

Carrying out the intigration by parts, Eq.(14) gives the expression:

$$\delta V=\iint\left[\left\{\frac{\partial A_2}{\partial\beta_1}T_2-\frac{\partial}{\partial\beta_1}(A_2T_1)-\frac{\partial A_1}{\partial\beta_2}S-\frac{\partial}{\partial\beta_2}(A_1S)-A_1A_2q_1\right\}\delta u_0\right.$$

$$+\left\{\frac{\partial A_1}{\partial \beta_2} T_1 - \frac{\partial}{\partial \beta_2}(A_1 T_2) - \frac{\partial A_2}{\partial \beta_1} S - \frac{\partial}{\partial \beta_1}(A_2 S) - A_1 A_2 q_2\right\} \delta v_0$$

$$+\left\{D_1 \nabla^2 \nabla^2 w + \frac{T_1}{R_1} + \frac{T_2}{R_2} + \nabla^2 \frac{M_T}{(1-\nu)} - qn\right\} \delta w \Big] d\beta_1 \, d\beta_2$$

$$+\int \Big[A_1\Big\{T_2 \delta v_0 + S \delta u_0 - \frac{M_2}{A_2} \frac{\partial \delta w}{\partial \beta_2} + \Big(N_2 + \frac{1}{A_1} \frac{\partial M_{21}}{\partial \beta_1}\Big) \delta w\Big\}\Big]_0^{\beta_2} d\beta_1$$

$$+\int \Big[A_2\Big\{T_1 \delta u_0 + S v_0 - \frac{M_1}{A_1} \frac{\partial \delta w}{\partial \beta_1} + \Big(N_1 + \frac{1}{A_2} \frac{\partial M_{21}}{\partial \beta_2}\Big) \delta w\Big\}\Big]_0^{\beta_1} d\beta_2$$

$$-2\,\big[M_{12}\, \delta w\big]_0^{\beta_1 \beta_2}$$

$$= 0 \qquad \text{..........(15)}$$

In the above expression, equating the coefficients of δu_0, δv_0 and δw equal to zero, gives the equilibrium equations.

$$\delta u_0 : \quad \frac{\partial}{\partial \beta_1}(A_2 T_1) - \frac{\partial A_2}{\partial \beta_1} T_2 + \frac{\partial A_1}{\partial \beta_2} S + \frac{\partial}{\partial \beta_2}(A_1 S) + A_1 A_2 q_1 = 0 \qquad \text{........(16)}$$

$$\delta v_0 : \quad \frac{\partial}{\partial \beta_2}(A_1 T_2) - \frac{\partial A_1}{\partial \beta_2} T_1 + \frac{\partial A_2}{\partial \beta_1} S + \frac{\partial}{\partial \beta_1}(A_2 S) + A_1 A_2 q_2 = 0 \qquad \text{.........(17)}$$

$$\delta w : \quad D_1 \nabla^2 \nabla^2 w + \frac{T_1}{R_1} + \frac{T_2}{R_2} = qn - \nabla^2 \frac{M_T}{(1-\nu)} \qquad \text{.........(18)}$$

where,

$$\nabla^2 (\ldots) = \frac{1}{A_1 A_2}\left\{ \frac{\partial}{\partial \beta_1}\left[\frac{A_2}{A_1} \frac{\partial(\ldots)}{\partial \beta_1} \right] + \frac{\partial}{\partial \beta_2}\left[\frac{A_1}{A_2} \frac{\partial(\ldots)}{\partial \beta_2} \right]\right\}$$

vanishing of surface integral terms in equation (15) give the appropriate boundry conditions considering the surface integrals, to get the conditions along $\beta_1 = 0$ and $\beta_1 = \bar{\beta}_1$, the surface integral must vanish.

$$\int A_2 \left\{ T_1\, \delta u_o + S\, \delta v_o - \frac{M_1}{A_1} \frac{\partial \delta w}{\partial \beta_1} + (N_1 + \frac{1}{A_2} \frac{\partial M_{12}}{\partial \beta_2})\, \delta w \right\} d\beta_2 = 0 \quad (19\text{-a})$$

2 This can be done by choosing

$$T_1\, \delta u_o = 0 \;; \qquad S\, \delta v_o = 0 \qquad (19\text{-b})$$

$$M_1 \frac{\partial \delta w}{\partial \beta_1} = 0 \;; \qquad (N_1 + \frac{1}{A_2} \frac{\partial M_{12}}{\partial \beta_2})\, \delta w = 0$$

where $N_1 + \frac{1}{A_2} \frac{\partial M_{12}}{\partial \beta_2}$ is called Kirchhoff's shear.

This physically means that the virtual work done by forces and moments along the boundries $\beta_1 = 0$ and $\beta_1 = \bar{\beta}_1$ must be zero. This can be done in three ways, firstly by making all forces and moments to be zero along the boundry or secondly by choosing the displacements and slopes along the boundry to be zero, or thirdly by choosing partly forces and moments to be zero and partly displacements and slopes to be zero along the boundry.

Similarly vanishing of surface integral for $\beta_2 = 0$ and $\beta_2 = \bar{\beta}_2$ is given

$$\int A_1 \{ T_2 \delta v_o + S \delta u_o - \frac{M_2}{A_2} \frac{\partial \delta w}{\partial \beta_2} +$$

$$(N_2 + \frac{1}{A_1} \frac{\partial M_{12}}{\partial \beta_1}) \delta w \} d\beta_1 = 0 \qquad (20\text{-a})$$

Boundary conditions along $\beta_2 = 0$ and $\beta_2 = \beta_2$ can be obtained as

$$T_2 \delta v_o = 0 \quad ; \quad S \delta u_o = 0$$

$$M_2 \frac{\partial \delta w}{\partial \beta_2} = 0 \quad ; \quad (N_2 + \frac{1}{A_1} \frac{\partial M_{12}}{\partial \beta_1}) \delta w = 0 \qquad (20\text{-b})$$

At the four corners,

$$2M_{12} \delta w = 0 \qquad (20\text{-c})$$

Consider a stress function 'θ' which is related to the inplane forces as follows:

$$T_1 = -\frac{1}{A_2} \frac{\partial}{\partial \beta_2} \left(\frac{1}{A_2} \frac{\partial \theta}{\partial \beta_2} \right) - \frac{1}{A_1^2 A_2} \frac{\partial A_2}{\partial \beta_1} \frac{\partial \theta}{\partial \beta_1} \qquad (21\text{-a})$$

$$T_2 = -\frac{1}{A_1} \frac{\partial}{\partial \beta_1} \left(\frac{1}{A_1} \frac{\partial \theta}{\partial \beta_1} \right) - \frac{1}{A_2^2 A_1} \frac{\partial A_1}{\partial \beta_2} \frac{\partial \theta}{\partial \beta_2} \qquad (21\text{-b})$$

$$S = \frac{1}{A_1 A_2} \left(\frac{\partial^2 \theta}{\partial \beta_1 \partial \beta_2} - \frac{1}{A_1} \frac{\partial A_1}{\partial \beta_2} \frac{\partial \theta}{\partial \beta_1} - \frac{1}{A_2} \frac{\partial A_2}{\partial \beta_1} \frac{\partial \theta}{\partial \beta_2} \right) \qquad (21\text{-c})$$

Putting the values of T_1, T_2 and S into eqns. (16), (17) and (18) and simplifying by using the Gauss-Codazzi relations[5].

$$\frac{1}{R_1 R_2} \frac{1}{A_1} \frac{\partial \theta}{\partial \beta_1} = q_1 \qquad (22)$$

$$\frac{1}{R_1 R_2} \frac{1}{A_2} \frac{\partial \theta}{\partial \beta_2} = q_2 \qquad (23)$$

$$D_1 \nabla^2 \nabla^2 w - L(\theta) = q_n - \nabla^2 \frac{M_T}{(1-\nu)} \tag{24}$$

$$L(\ldots) = \frac{1}{A_1A_2}\left\{ \frac{\partial}{\partial\beta_1}\left[\frac{1}{R_2} \frac{A_2}{A_1} \frac{\partial(\ldots)}{\partial\beta_1}\right] + \right.$$

$$\left. + \frac{\partial}{\partial\beta_2}\left[\frac{1}{R_1} \frac{A_1}{A_2} \frac{\partial(\ldots)}{\partial\beta_2}\right]\right\}$$

In absence of forces in β_1 and β_2 directions, Eqns. (22) and (23) will be automatically satisfied for shell of zero Gaussian curvature, for shell of non zero Gaussian curvature, the above equations are approximately satisfied for shallow shells. In order to obtain, the unknown stress function 'θ', now we have to consider the compatibility conditions.

Compatibility Equation is given by

$$\frac{k_1}{R_2} + \frac{k_2}{R_1} + \frac{1}{A_1A_2}\left[\frac{\partial}{\partial\beta_1} \frac{1}{A_1}\left\{ A_2 \frac{\partial e_2^o}{\partial\beta_1} + \frac{\partial A_2}{\partial\beta_1}(e_2^o - e_1^o) - \right.\right.$$

$$\left. \frac{1}{2} A_1 \frac{\partial e_{12}^o}{\partial\beta_2} - \frac{\partial A_1}{\partial\beta_2} e_{12}^o \right\} + \frac{\partial}{\partial\beta_2} \frac{1}{A_2}\left\{ A_1 \frac{\partial e_1^o}{\partial\beta_2} + \right.$$

$$\left.\left. \frac{\partial A_1}{\partial\beta_2}(e_1^o - e_2^o) - \frac{1}{2} A_2 \frac{\partial e_{12}^o}{\partial\beta_1} - \frac{\partial A_2}{\partial\beta_1} e_{12}^o \right\}\right] = 0 \tag{25}$$

By substituting the values of k_1, k_2, e_1^o, e_2^o, e_{12}^o in Eqn. (25).

$$EhL(w) + \nabla^2 \nabla^2\theta + \frac{(1+\nu)}{A_1A_2}\left[\frac{\partial}{\partial\beta_1}\left(\frac{1}{R_1}\frac{A_2}{A_1}\frac{1}{R_2}\frac{\partial\theta}{\partial\beta_1}\right) + \right.$$

$$\left. \frac{\partial}{\partial\beta_2}\left(\frac{1}{R_1R_2} \frac{A_1}{A_2} \frac{\partial\theta}{\partial\beta_2}\right)\right] = \nabla^2 N_T \tag{26}$$

for zero Gaussian curvature ($1/R_1R_2 = 0$)

$$EhL(w) + \nabla^2 \nabla^2\theta = \nabla^2 N_T \tag{27}$$

These equations hold good for

1. shells of zero Gaussian curvature such as cylindrical shells and cones.
2. good for shallow shells
3. approximate for non-zero Gaussian curvature.

These equations are coupled and requires the simultaneous solution of deflection 'w' and stress function 'Ø', If there is no external loading, for constant thickness equation (24) reduces to the form,

$$D_1 \nabla^2 \nabla^2 w - L(\emptyset) = - \nabla^2 \frac{M_T}{(1-\nu)} \qquad (28)$$

Superposition of deflections due to transverse load alone and deflection due to temperature distribution alone is possible. Solution of deflection due to temperature distribution alone has been utilised in the present case.

CHAPTER III

METHOD OF SOLUTIONS

In this chapter, solutions to obtain thermal stresses in thin shallow shells with two edges simply and other two edges supported in any manner are developed. The equations governing, the transverse diffection of elastic thin shells subjected to a temperature distribution which is anti-symmetric about the middle plane and otherwise arbitrary are:

$$D_1 \nabla^2\nabla^2 w - L(\theta) + \nabla^2 \frac{M_T}{(1-\nu)} = 0 \qquad (29)$$

$$EhL(w) + \nabla^2\nabla^2\theta - \nabla^2 N_T = 0 \qquad (30)$$

Deflection function 'w' and stress function 'θ' may be seperated out as

$$D_1 \nabla^8\theta + EhL^2(\theta) = D_1\nabla^6 N_T + Eh\nabla^2 L\left(\frac{M_T}{(1-\nu)}\right) \qquad (31)$$

$$D_1 \nabla^8 w + EhL^2(w) = -\nabla^6 \frac{M_T}{(1-\nu)} + \nabla^2 L(N_T) \qquad (32)$$

Both functions w and θ, have the same homogeneous solutions but differing in constants of integrations.

Doubly curved thin shells may be reduced to thin shallow shells, when it is satisfying the following conditions

$$Z = Z(x, y) \quad , \quad H < \frac{1}{5}(a, b)$$

$$\left(\frac{\partial Z}{\partial x}\right)^2 \ll 1 \quad , \quad \left(\frac{\partial Z}{\partial y}\right)^2 \ll 1, \quad \left(\frac{\partial Z}{\partial x} \cdot \frac{\partial Z}{\partial y}\right) \ll 1$$

Thin shallow shell has been shown in fig. (2) with coordinate system. The value of A_1 and A_2 for shallow shell may be approximated as

$A_1 = A_2 = 1$, R_1 and R_2 are constant.

Stress couples and stress resultants for thin shallow shell in terms of deflection 'w' and stress function θ may be expressed as

$$M_x = - D_1 \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) - \frac{M_T}{(1-\nu)} \quad (33)$$

$$M_y = + D_1 \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) - \frac{M_T}{(1-\nu)} \quad (34)$$

$$M_{xy} = + D_1(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \quad (35)$$

$$T_1 = - \frac{\partial^2 \theta}{\partial y^2}, \quad T_2 = - \frac{\partial^2 \theta}{\partial x^2}, \quad S = \frac{\partial^2 \theta}{\partial x \partial y} \quad (36\text{-a-c})$$

The middle plane strains in terms of stress resultants may be expressed as,

$$e_1^o = \frac{1}{Eh} (T_1 - \nu T_2 + N_T), \quad e_2^o = \frac{1}{Eh} (T_2 - \nu T_1 + N_T)$$

$$(37\text{-a-b})$$

Doubly curved thin shallow shell is simply supported along x = 0 and x = a and making conditions such that

$$T_1 \left(\begin{smallmatrix}0\\a\end{smallmatrix}\right) = 0 \quad \text{and} \quad e_2^o \left(\begin{smallmatrix}0\\a\end{smallmatrix}\right) = 0 \text{ (using tie rod)}$$

$$T_1 \left(\begin{smallmatrix}0\\a\end{smallmatrix}, y\right) = - \frac{\partial^2}{\partial y^2} \theta\left(\begin{smallmatrix}0\\a\end{smallmatrix}, y\right) = 0$$

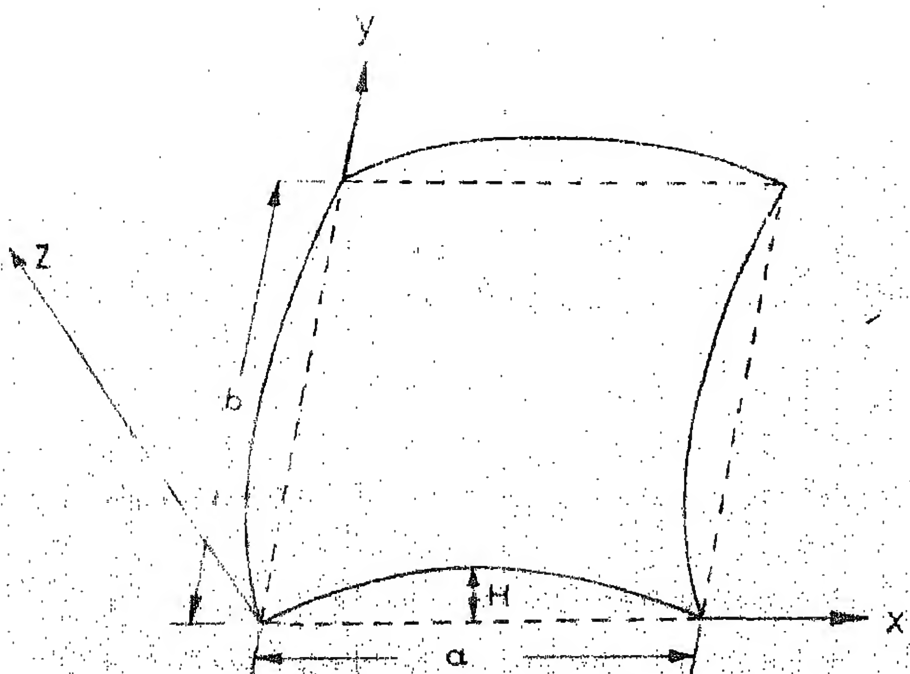

FIG.2 SHALLOW SHELL WITH COORDINATES

or

$$\theta(^{o}_{a}, y) = 0 \qquad (38)$$

Distribution of temperature is antisymmetric about the middle plane so that θ_y $(^{o}_{a}) = 0$

$$\theta^{o}_{x}(^{o}_{a}, y) = \frac{\partial^2 \theta}{\partial x^2}\Big|_{x = o, a} = 0$$

or

$$\frac{\partial^2 \theta}{\partial x^2}(^{o}_{a}, y) = 0 \qquad (39)$$

$$w = 0; \quad M_x = 0 \quad \text{when} \quad x = o,\ a \qquad (40)$$

Boundary conditions (40) imply that

$$w = 0; \quad \frac{\partial^2 w}{\partial x^2} = -M_T(0,y)/D_1(1-\nu) \text{ along } x = 0$$

$$w = 0; \quad \frac{\partial^2 w}{\partial x^2} = -M_T(a,y)/D_1(1-\nu) \text{ along } x = a$$

The boundary conditions are non-homogeneous. The non-homogeneity is removed by taking the function 'w' into two forms[6,7,8] as follows

$$w(x, y) = U(x, y) + \left\{\frac{M_T(0,y)}{(1-\nu)} H_o(x) + \frac{M_T(a,y)}{(1-\nu)} H_a(x)\right\} \qquad (41)$$

where $H_o(x)$ and $H_a(x)$ are the unknown functions of x only. Applying the boundary conditions into the equation (41)

$$U(0, y) = -\left[\frac{M_T(0, y)}{(1-\nu)} H_o(0) + \frac{M_T(a, y)}{(1-\nu)} H_a(0)\right] \qquad (42\text{-a})$$

$$\frac{\partial^2}{\partial x^2} U(0, y) = -\frac{M_T}{(1-\nu)}(0, y)\left[\frac{1}{D_1} + \frac{d^2}{dx^2} H_o(0)\right] - \frac{M_T(a,y)}{(1-\nu)}\frac{d^2}{dx^2} H_a(0)$$

$$(42\text{-b})$$

$$U(0,y) = -\left[\frac{M_T(0,y)}{(1-\nu)} H_o(a) + \frac{M_T(a,y)}{(1-\nu)} H_a(a)\right] \qquad (42\text{-}c)$$

$$\frac{\partial^2}{\partial x^2} U(a,y) = -\frac{M_T(0,y)}{(1-\nu)} \frac{d^2}{dx^2} H_o(a) - \frac{M_T(a,y)}{(1-\nu)} \left[\frac{1}{D_1} + \frac{d^2}{dx^2} H_a(a)\right]$$

$$\ldots \quad (42\text{-}d)$$

Functions $H_o(x)$ and $H_a(x)$ are chosen in the following manner in order to make the equation (42) homogeneous.

$$H_o(0) = 0 \qquad\qquad H_a(0) = 0$$

$$\frac{d^2}{dx^2} H_o(0) = -1/D_1 \qquad\qquad \frac{d^2}{dx^2} H_a(0) = 0$$

$$(43\text{-}a) \qquad\qquad (43\text{-}b)$$

$$H_o(a) = 0 \qquad\qquad H_a(a) = 0$$

$$\frac{d^2}{dx^2} H_o(a) = 0 \qquad\qquad \frac{d^2}{dx^2} H_a(a) = -1/D_1$$

The functions $H_o(x)$ and $H_a(x)$ are taken as polynomials with four constants which are determined with above conditions on $H_o(x)$ and $H_a(x)$. They are of the form

$$H_o(x) = \frac{a^2}{D_1}\left[\frac{1}{3}\frac{x}{a} - \frac{1}{2}\frac{x^2}{a^2} + \frac{1}{6}\frac{x^3}{a^3}\right] \qquad (44\text{-}a)$$

$$H_a(x) = \frac{a^2}{6D_1}\left[x/a - \frac{x^3}{a^3}\right] \qquad (44\text{-}b)$$

The boundary conditions on $U(x, y)$ become,

$$U = 0, \quad \frac{\partial^2 U}{\partial x^2} = 0 \quad \text{along } x = 0 \text{ and } x = a.$$

Now substituting the eqn. (41) in eqn. (32)

$$D_1 \nabla^8 U + EhL^2(U) = F(x,y) \tag{45-a}$$

where $$F(x,y) = -\nabla^6 \frac{M_T}{(1-\nu)} + \nabla^2 L(M_T) - D_1\nabla^8 \left[\frac{M_T}{(1-\nu)}(0,y) H_0(x) + \frac{M_T}{(1-\nu)}(a,y) H_a(x) \right] - EhL^2 \left[\frac{M_T}{(1-\nu)}(0,y) H_0(x) + \frac{M_T}{(1-\nu)}(a,y) H_a(x) \right] \tag{45-b}$$

Solution of eqn. (45-a) may be taken as

$$U(x,y) = \sum_{m=1}^{\infty} Y_m(y) \sin \frac{m\pi x}{a} \tag{46}$$

This form of solution will satisfy the boundary conditions along x = 0 and x = a.

Substituting the value of U(x, y) in the equation 45(a) and simplifying

$$\left(\frac{d^2}{dy^2} Y_m(y) - \frac{m^2\pi^2}{a^2} Y_m(y) \right)^4 + \frac{Eh}{D_1} \left(\frac{1}{R_1} \frac{d^2}{dy^2} Y_m(y) - \frac{1}{R_2} \frac{m^2\pi^2}{a^2} Y_m(y) \right)^2 = K_m(y) \tag{47-a}$$

where $$K_m(y) = \frac{1}{D_1} \frac{2}{a} \int_0^a f(x,y) \sin m\pi x/a \, dx$$

or $$A_{1m} \frac{d^8}{dy^8} Y_m(y) + A_{2m} \frac{d^6}{dy^6} Y_m(y) + A_{3m} \frac{d^4}{dy^4} Y_m(y) + A_{4m} \frac{d^2}{dy^2} Y_m(y) + A_{5m} Y_m(y) = K_m(y) \tag{47-b}$$

<u>Solution of homogeneous equation:</u>

$$A_{1m} \frac{d^8}{dy^8} Y_m(y) + A_{2m} \frac{d^6}{dy^6} Y_m(y) + A_{3m} \frac{d^4}{dy^4} Y_m(y) + A_{4m} \frac{d^2}{dy^2} Y_m(y) + A_{5m} Y_m(y) = 0 \tag{47-c}$$

Assume $Y_m(y)$ in the form:

$$Y_m(y) = C_m e^{\delta_m y/a}$$

Substituting value of $Y_m(y)$ into the equation (47-a), gives,

$$B_{1m} \delta_{mm}^8 + B_{2m} \delta_m^6 + B_{3m} \delta_m^4 + B_{4m} \delta_m^2 + B_{5m} = 0 \qquad (49\text{-a})$$

where
$$\begin{array}{lll} B_{1m} = 1 & A_m = B_{1m} & \\ B_{2m} = -4 m^2\pi^2 & A_{2m} = B_{2m}/a^2 & \\ B_{3m} = (6 m^4\pi^4 + T) & A_{3m} = B_{3m}/a^4 & \\ B_{4m} = -2m^2\pi^2(2m^4\pi^4 + TR) & A_{4m} = B_{4m}/a^6 & (49\text{-b}) \\ B_{5m} = m^4\pi^4(m^4\pi^4 + R^2T) & A_{5m} = B_{5m}/a^8 & \end{array}$$

$$T = 12(1-\nu)^2 \frac{a^4}{h^2R_1^2}, \qquad R = R_1/R_2$$

Roots of the equation (49) may be obtained by changing this equation into fourth order equation and solving by the method given in Ref. (9).

Depending upon the values of A_{1m} to A_{5m}, eqn. (47-c) has different set of roots and different type of function $Y_m(y)$ as given in the Appendix (A).

The homogeneous solution[10] of eqn. (47) may be expressed as

$$Y_m(y) = C_{1m} y_{1m} + C_{2m} y_{2m} + \dots\dots\dots + C_{8m} y_{8m} \qquad (50)$$

where y_{1m} through y_{8m} are the basis of solutions (Appendix A) and C_{1m} through C_{8m} are arbitrary constants to be evaluated from the boundary conditions of the edges of thin shallow shell

along $y = 0$ and $y = b$.

Writing the equation (50) in the matrix forms as

$$Y_m(y) = [C]^T \; [Y_R(y)] \qquad (51\text{-}1)$$

The superscript 'T' denotes the transpose of the matrix and

$$[C] = \begin{bmatrix} C_{1m} \\ C_{2m} \\ C_{3m} \\ C_{4m} \\ C_{5m} \\ C_{6m} \\ C_{7m} \\ C_{8m} \end{bmatrix} \qquad [y_R(y)] = \begin{bmatrix} y_{1m}(y) \\ y_{2m}(y) \\ y_{3m}(y) \\ y_{4m}(y) \\ y_{5m}(y) \\ y_{6m}(y) \\ y_{7m}(y) \\ y_{8m}(y) \end{bmatrix} \qquad (51\text{-}b\text{-}c)$$

The differential equation of $[y_R(y)]$ can be expressed as

$$\frac{d^n}{dy^n} [Y_R(y)] = [N_R]^n \; [Y_R(y)] \qquad (51)$$

$$[N_R]^n = [N_R][N_R]\ldots\ldots\ldots \; n \text{ Times} \qquad (52)$$

where $[N_R]$ is a matrix depending on the characteristic roots of the differential equation (47) and is given in Appendix A.

$$\frac{d^n}{dy^n} Y_m(y) = [C]^T [N_R]^n [Y_R(y)] \qquad (53)$$

From equation (51) it is clear that function $\emptyset$ has the identical homogeneous solution to the diflection function 'w' except integration constants.

For homogeneous boundary conditions at x = 0, and x = a

$$\theta(x,y) = \sum_{m=1,3}^{\infty} F_m'(y) \sin m\pi x/a \tag{54}$$

$$F_m(y) = [D]^T [Y_R(y)] \tag{55}$$

$$[D] = \begin{bmatrix} D_{1m} \\ D_{2m} \\ D_{3m} \\ D_{4m} \\ D_{5m} \\ D_{6m} \\ D_{7m} \\ D_{8m} \end{bmatrix}$$

There are sixteen unknowns, eight in deflection 'w' and eight in stress function 'θ'. But possible number of boundary conditions are eight along the edges y = 0 and y = b. Eight unknowns can be expressed in terms of other eight unknowns as given in reference (11).

Substitute the value of θ(x,y) and homogeneous boundary condition part of w(x,y) in the equation (30)

$$Eh\left\{-\frac{1}{R_2}\frac{m^2\pi^2}{a^2} + \frac{1}{R_1}[N_R]^2\right\}[C]^T[Y_R(y)] + \left\{\frac{m^4\pi^4}{a^4} + [N_R]^4 - \frac{m^2\pi^2}{a^2}[N_R]^2\right\}[D]^T[Y_R(y)] = 0 \quad ..(56)$$

To satisfy this equation, equal coefficient of y_{1m} through y_{8m} equal to zero. This gives eight equations relating the constants C_{1m}, C_{2m}, C_{3m}, C_{4m}, C_{5m}, C_{6m}, C_{7m}, C_{8m} to D_{1m}, D_{2m}, D_{3m}, D_{4m}, D_{5m}, D_{6m}, D_{7m}, D_{8m}. Expressing the D_{1m} through D_{8m} constants into C_{1m} through C_{8m}. Thus Z_{1m} through Z_{8m} are known in terms of C_{1m} through C_{8m}. And there are eight unknowns C_{1m} through C_{8m}.

<u>Solution of non-homogeneous equation by operator:</u>

Equation (47-b) can be written as

$$(D-a_1)\,(D-a_2)\ \ldots\ (D-a_8) = K(y) \qquad (57)$$

where a_1, a_2 are the robts of the auxielly equation

Put $\quad u_1 = (D-a_2)\,(D-a_3)\ \ldots\ (D-a_8)\ Y_m(y)$

So that equation (57) becomes

$$(D-a_1)\ u_1 = K(y)$$

This is a first-order linear equation in the unknown u_1, for which a particular integral is readly found. Next put

$$u_2 = (D-a_3)\,(D-a_4)\ \ldots\ (D-a_8)\ Y_m(y)$$

and write

$$(D-a_2)\ u_2 = u_1$$

where u_1 is known function of y, and solve this first order equation for u_2 continuing in this way, a sequence of linear first order equations are obtained.

$$(D-a_i)\ u_i = u_{i-1} \qquad (i = 1, 2, \ldots 8) \qquad (58)$$

where

$$u_0 = K(y)\ ,\quad u_8 = Y_m(y),\quad u_i = (D-a_{i+1})(D-a_{i+2})$$
$$\ldots(D-a_8)\ Y_m(y) \qquad (i = 1,2,\ \ldots 7)$$

and where, for each value of i, u_{i-1} is a known function of y. After seven such steps, obtain the final first order differential equation of the sequence (58), namely

$$(D-a_8)\ Y_m(y) = u_7$$

a solution of which is a particular integral of (57). The sequential integration of the equation (58) is obviously a routine process. Each such equation has the integrating factor $e^{a_i y}$ and the solution

$$u_i = e^{a_i y} \int e^{-a_i y}\ u_{i-1}(y)\ dy \quad (i = 1, \ldots 8)$$

The unknown constants C_{1m} through C_{8m} may be obtained by applying the suitable eight initial parameters, displacements, slopes, forces, and moments in terms of $Y_m(y)$, $Y_m'(y)$, $Y_m''(y)$, $F_m(y)$, $F_m'(y)$ and $F_m''(y)$ along the edges y = 0 and y = b, depending upon the boundary conditions at y = 0 and y = b.

This may be expressed in matrix form as

$$[I_{pm}] = [A]\ [C] \tag{59}$$

$$[C] = [A]^{-1}\ [I_{pm}] \tag{60}$$

The value of I_{pm} matrix depends upon the boundary conditions. Values of I_{pm} matrix have been given for simply supported along y = 0 and y = b and clamped along y = 0 and y = b.

Matrix [A] is given as

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} & A_{28} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} & A_{37} & A_{38} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} & A_{47} & A_{48} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} & A_{57} & A_{58} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} & A_{67} & A_{68} \\ A_{71} & A_{72} & A_{73} & A_{74} & A_{75} & A_{76} & A_{77} & A_{78} \\ A_{81} & A_{82} & A_{83} & A_{84} & A_{85} & A_{86} & A_{87} & A_{88} \end{bmatrix}$$

Value of Matrix [A] depends upon $[Y_m(y)]$ and type of boundary conditions along $y = 0$ and $y = b$.

Values of elements of Matrix [A] have been given in example for simply supported along $y = 0$ and $y = b$.

For the sake of simplicity, take temperature distribution independent of x and y Eqn. (47-b) reduces to homogeneous form. So deflection function may be given by

$$w(x, y) = \sum_{m=1}^{\infty} \left[Y_m(y) + 4 \frac{M_T \; a^2}{D_1(1-\nu)m^3\pi^3} \right] \sin m\pi x/a$$

(a) <u>Simply supported along the edges y = 0 and y = b</u>:

The boundary conditions are,

$w(x, {0 \atop b}) = 0.$ This gives $Y_m(0) = Y_m(b) = -\dfrac{4M_T \; a^2}{D(1-\nu)m^3\pi^3}$

for m odd only

$M_y(x, {}^{0}_{b}) = 0$

This gives $\frac{d^2}{dy^2} Y_m(0) = \frac{d^2}{dy^2} Y_m(b) = - \frac{4M_T}{(1-\nu) D_1 \pi m}$

for m odd only

$Z_2(x, {}^{0}_{b}) = 0$ this gives $F_m({}^{0}_{b}) = 0$

$U(x, {}^{0}_{b}) = 0$ This gives $\frac{d^2}{dy^2} X_m({}^{0}_{b}) = 0.$

$$[I_{1m}] = \begin{bmatrix} \frac{-4 M_T a^2}{D_1(1-\nu) m^3\pi^3} \\ \frac{-4 M_T a^2}{D_1(1-\nu) m^3\pi^3} \\ -4 M_T / D_1(1-\nu) m\pi \\ -4 M_T / D_1(1-\nu) m\pi \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) <u>Fixed along the edges y = 0 and y = b</u>

Boundary conditions are

$W(x, {}^{0}_{b}) = 0$

This gives $Y_m(0) = Y_m(b) = \frac{4 M_T a^2}{(1-\nu) D m^3 \pi^3}$ for m odd only

$$\frac{\partial}{\partial y} W(x, {}^{o}_{b}) = 0$$

This gives $\frac{d}{dy} Y_m(0) = \frac{d}{dy} Y_m(b) = 0$

$T_2(x, {}^{o}_{b}) = 0$, this gives $F_m({}^{o}_{b}) = 0$

$U(x, {}^{o}_{b}) = 0$, this gives $\frac{d^2}{dy^2} F_m({}^{o}_{b}) = 0.$

$$[I_{pm}] = \begin{bmatrix} \dfrac{-4M_T a^2}{D_1(1-\nu)m^3\pi^3} \\ -4M_T a^2/D_1(1-\nu)m^3\pi^3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Y_m(y) = [C]^T [Y_R(y)]$$

$$W(x, y) = \sum_{m=1,3}^{\infty} \left\{ Y_m(y) + \frac{4 M_T a^2}{D_1(1-\nu) m^3\pi^3} \right\} \sin m\pi x/a \quad (61)$$

$$M_x = \sum_{m=1,3}^{\infty} \left\{ D_1 \frac{m^2\pi^2}{a^2} [C]^T + [C]^T [N_R]^2 \right\} [Y_R(y)] \sin m\pi \quad (62)$$

$$M_y = \sum_{m=1}^{\infty} \left\{ D_1 \left\{ -[C]^T [N_R]^2 + \frac{m^2\pi^2}{a^2} [C]^T \right\} [Y_R(y)] - \frac{4M_T}{(1-\nu) m\pi} \right\} \sin m\pi x/a \quad (63)$$

$$M_{xy} = + D_1(1-\nu) \Sigma \frac{m\pi}{a} [C]^T [\,^s R] [Y_R(y)] \cos m\pi x/a \quad (64)$$

(c) Example:

In order to illustrate the method, an example with $a^2/4R_1=5$ $\nu = .3$, and $R_1/R_2 = 0.5$ has been worked out below.

Roots of the equation (43-a) have the form as $\pm n_1$, $\pm n_2$ $\pm (n_3 \pm in_4)$

$$Y_R(y)^T = \{\cosh n_1 y/a,\ \sin n_1 y/a,\ \cosh n_2 y/a,\ \sinh n_2 y/a,$$
$$\cosh n_3 y/a \cos n_4 y/a\ ,\ \sinh n_3 y/a \cos n_4 y/a,$$
$$\cosh n_3 y/a \sin n_4 y/a,\ \sinh n_3 y/a \sin n_4 y/a\}$$

Equating the coefficients of elements of matrix $[Y_R(y)]^T$ in equation (56) to zero, constants D_{1m} to D_{8m} are obtained interms of C_{1m} to C_{8m} as

$$D_{1m} = KP_9 \ C_{1m} \ , \quad D_{2m} = KP_9 \ C_{2m}$$

$$D_{3m} = KP_{10} \ C_{3m} \ , \quad D_{4m} = KP_{10} \ C_{4m}$$

$$D_{4m} = K(P_7 C_{5m} + P_8 \ C_{8m}), \quad D_{6m} = K(P_7 \ C_{6m} + P_8 \ C_{7m})$$

$$D_{7m} = K(P_5 \ C_{6m} + P_6 \ C_{7m}), \quad D_{8m} = K(P_5 C_{5m} + P_6 C_{8m})$$

where

$$P_1 = - R_1/R_2 \ m^2\pi^2 + n_3^2 - n_4^2$$

$$P_2 = 2 \, n_3 \, n_4$$

$$P_3 = m^4\pi^4 + n_3^4 + n_4^4 - 6n_3^2 - m^2\pi^2 (n_3^2 - n_4^2)$$

$$P_4 = 4n_3^3 \, n_4 - 4n_4^3 \, n_3 - 2m^2\pi^2 \, n_3 n_4$$

$P_5 = (P_4P_1 - P_3P_2) / P$

$P_6 = (P_2P_4 + P_1P_3)/P$

$P_7 = (P_1P_3 + P_2P_4)/P$

$P_8 = (P_2P_3 - P_1P_4)/P$

$P_9 = (-R_1/R_2\ m^2\pi^2 + n_1^2) / (m^4\pi^4 + n_1^4 - m^2\pi^2\ n_1^2)$

$P_{10} = (-R_1/R_2\ m^2\pi^2 + n_2^2) / (m^4\pi^4 + n_2^4 - m^2\pi^2\ n_2^2)$

$K = -\dfrac{Eh\ a^2}{R_1} \qquad P = P_3^2 + P_4^2$

Matrix [A] has different values for different types of boundary conditions.

<u>Simply supported</u> along $y = 0$ and $y = b$

$A_{11} = A_{13} = A_{15} = 1,$ $\qquad A_{12} = A_{14} = A_{16} = A_{17} = A_{18} = 0$

$A_{21} = \text{Cosh}\ n_1b/a$ $\qquad A_{22} = \text{Sinh}\ n_1b/a$

$A_{23} = \cosh\ n_2b/a$ $\qquad A_{24} = \text{Sinh}\ n_2b/a$

$A_{25} = \cosh\ n_3b/a\ \cos\ n_4b/a$ $\qquad A_{26} = \sinh\ n_3b/a\ \cos\ n_4b/a$

$A_{27} = \cos\ n_3b/a\ \sin\ n_4b/a$ $\qquad A_{28} = \sinh\ n_3b/a\ \sin\ n_4b/a$

$A_{31} = n_1^2$ $\qquad A_{33} = n_2^2$

$A_{35} = n_3^2 - n_4^2$ $\qquad A_{38} = 2n_3n_4$

$A_{32} = A_{34} = 0$ $\qquad A_{36} = A_{37} = 0$

$A_{41} = n_1^2\ A_{21}$ $\qquad A_{42} = n_1^2\ A_{22},$

$A_{43} = n_2^2\ A_{23}$ $\qquad A_{44} = n_2^2\ A_{24}$

$A_{46} = A_{35}\,A_{26} - P_2\,A_{27}$ $\qquad$ $A_{45} = A_{35}\,A_{25} - P_2\,A_{28}$

$A_{47} = A_{35} + P_2\,A_{26}$ $\qquad$ $A_{48} = A_{35}\,A_{28} + P_2\,A_{25}$

$A_{51} = P_9$. $\qquad$ $A_{53} = P_{10}$

$A_{55} = P_7$ $\qquad$ $A_{58} = P_8$

$A_{52} = A_{54} = 0$ $\qquad$ $A_{56} = A_{57} = 0$

$A_{61} = P_9\,A_{21}$ $\qquad$ $A_{62} = P_9\,A_{22}$

$A_{63} = P_{10}\,A_{23}$ $\qquad$ $A_{64} = P_{10}\,A_{24}$

$A_{65} = P_7\,A_{25} + P_5\,A_{28}$ $\qquad$ $A_{66} = P_7\,A_{26} + P_5\,A_{27}$

$A_{67} = P_8\,A_{26} + P_6\,A_{27}$ $\qquad$ $A_{68} = P_8\,A_{25} + P_6\,A_{28}$

$A_{71} = P_9\,n_1^2$ $\qquad$ $A_{73} = P_{10}\,n_2^2$

$A_{75} = P_7\,A_{35} + P_2\,P_5$ $\qquad$ $A_{78} = P_8\,A_{35} + P_2 P_6$

$A_{72} = A_{74} = 0$ $\qquad$ $A_{76} = A_{77} = 0$

$A_{81} = P_9\,A_{41}$ $\qquad$ $A_{82} = P_9\,A_{42}$

$A_{83} = P_{10}\,A_{43}$ $\qquad$ $A_{84} = P_{10}\,A_{44}$

$A_{85} = P_7\,A_{45} + P_5\,A_{48}$ $\qquad$ $A_{86} = P_7\,A_{46} + P_5\,A_{46}$

$A_{87} = P_8\,A_{46} + P_6\,A_{47}$ $\qquad$ $A_{88} = P_8\,A_{45} + P_6\,A_{48}$

Solution of the matrix equation (60), by putting the values of matrix $[I_{pm}]$ and matrix $[A]$, gives the values of constants, C_{1m} through C_{8m}. Deflection and moments may be obtained by substitution of C_{1m} through C_{8m} in Eqn. (61), (62, (63) and (64). Values of elements of the matrix $[A]$, for shallow shell with two edges simply supported and remaining two edges clamped, have been given in computer programme in Appendix C.

(d) Conclusions:

The method developed is more general than Ambartsumyan's[12] method. It does not require to define deflection in term of any other functions such as $w = \nabla^4 \psi_0$ where ψ_0 is any arbitrary function. Stress function 'θ' and deflection function 'w' are replaced in term of ψ_0 function in Eq. (29) in Ambartsumyan's method.

The solution of differential equation which is in term of ψ_0 function, is obtained with the help of boundary conditions. Then stress function and deflection 'w' are obtained from ψ_0 function. But the solution of differential equation which is in term of ψ_0 function is mathematically difficult or most probably impossible for cases where $\nabla^2 N_T$ is not zero. Method developed in preceeding sections may be applied for cases where $\nabla^2 N_T$ is not zero.

Present method can be used for any type of temperature distribution in their shallow shell. In such cases, the same method holds good except for addition of particular integral term in Eq. (50).

Parameter $a^2/h\ R_1$ and b/a ratio play an important role in magnitude and direction of deflection of thin shallow shell for fixed value of b/a ratio, dimensionless quantity for diflection of the shell decreases from upward direction to down-ward direction as values of the parameter $a^2/h\ R_1$ increases as shown in Appendix 'E', for all edges simply supported and

two edges simply supported and remaining two edges clamped. Similarly, for fixed value of parameter $a^2/h\ R_1$, dimensionless quantity for diflection of the shallow shell decreases from upward direction to down ward direction as values of the b/a ratio increases as shown in Appendix "." for all edges simply supported and two edges simply supported and remaining two edges clamped. Values of dimensionless quantities for M_x and M_y decreases in magnitude with increasing the parameter $a^2/h\ R_1$ and b/a ratio. But change in magnitude due to parameter a^2/hR_1 and b/a ratio of dimensionless quantities of moments M_x and M_y is small in comparison to the change in magnitude and direction of dimensionless quantity for deflection. Magnitude and direction of the dimensionless quantity for moment M_{xy} also change with change of parameter $a^2/h\ R_1$ and b/a ratio as shown in Appendix E. For all cases numerical values and graphs have been presented for deflection and moments (M_x and M_y) at $x = a/2$ and $R_1/R_2 = 0.5$ for vari various values of parameter a^2/hR_1 and b/a ratio. This has been done for parameter a^2/hR_1 ranging from 5.0 to 20.0 at intervals of 5.0, and b/a ratio wranging from 1.0 to 2.0 at interval of 0.2. The deflection and moments have been presented by the dimensionless parameters $wh/\alpha Ta^2$, $M_x/E\alpha Th^2$, $M_y/E\alpha Th^2$ and $M_{xy}/E\alpha Th^2$. Dimensionless Parameter $M_{xy}/E\alpha Th^2$ has been presented along $x = 0$. Only magnitude of dimensionless parameters $M_x/E\alpha Th^2$, $M_y/E\alpha Th^2$ have been presented in Appendix E and direction is given in Table Appendix D. Since the variation of $wh/\alpha Ta^2$, $M_x/E\alpha Th^2$, $M_y/E\alpha Th^2$ along x-axis is in simple sine form, the values of these quantities at any point can be easily calculated.

The results obtained in preceeding sections, may be used in design of shallow shell's roofs where outside the building, atmospheric temperature is different from inside temperature of the building. The method developed in preceeding sections may be used in finding out the stresses in end caps of boilers and airwings where temperature gradient exists, due to temperature difference of inside and outside surfaces.

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APPENDIX A

CHARACTERISTIC ROOTS OF THE DIFFERENTIAL EQUATION (47)

A.1 The homogeneous part of the differential equation (47-b) can be written as

$$A_{1m} \frac{d^8}{dy^8} Y_m(y) - A_{2m} \frac{d^6}{dy^6} + A_{3m} \frac{d^4}{dy^4} Y_m(y) - A_{4m} \frac{d^2}{dy^2} Y_m(y)$$

$$+ A_{5m} Y_m(y) = 0 \qquad (A\text{-}1)$$

where

A_{1m} to A_{5m} are given by eqn. (47-b)

The nature of the eight characteristic roots of Eq. (A-1) depends on the value of A_{1m} to A_{5m} and can be classified under the following cases.

Case 1 $\pm n_1, \quad \pm n_2, \quad \pm n_3, \quad \pm n_4$

Case 2 $\pm n_1, \quad \pm n_2, \quad \pm n_3, \quad \pm in_4$

Case 3 $\pm n_1, \quad \pm n_2, \quad \pm in_3, \pm in_4$

Case 4 $\pm n_1, \quad \pm in_2, \pm in_3, \pm in_4$

Case 5 $\pm in_1, \pm in_2, \pm in_3, \pm in_4$

Case 6 $\pm n_1, \quad \pm n_2, \quad \pm (n_3 \pm in_4)$

Case 7 $\pm n_1, \quad \pm in_2, \pm (n_3 \pm in_4)$

Case 8 $\pm in_1, \pm in_2, \pm (n_3 \pm in_4)$

Case 9 $\pm (n_1 \pm in_2), \quad \pm (n_3 \pm in_4)$

where n_1, n_2, n_3 and n_4 are the components of the characteristic roots.

A.2 Basis of Solutions:

The basis of solutions $[Y_R(y)]$, (Eq. (51-c)) is given below as a row matrix $[Y_R(y)]^T$, along with the matrix $[N_R]$, (Eq. (52)), for various cases listed in Eqns. (A-2)

Case 1

$$[Y_R(y)]^T = \{\cosh n_1y ,\ \sinh n_1y ,\ \cosh n_2y,\ \sinh n_2y \quad \cosh n_3y ,\ \sinh n_3y ,\ \cosh n_4y,\ \sinh n_4y\} \quad \text{(A-3a)}$$

$$[N_R] = \begin{bmatrix} 0 & d_1n_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ n_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2n_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_3n_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_4n_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_4 & 0 \end{bmatrix} \quad \text{(A-3b)}$$

where $d_1 = d_2 = d_3 = d_4 = 1$

Case 2

$$[Y_R(y)]^T = \{\cosh n_1y,\ \sinh n_1y ,\ \cosh n_2y ,\ \sinh n_2y, \quad \cosh n_3y,\ \sinh n_3y,\ \cos n_4y ,\ \sin n_4y\} \quad \text{(A-4a)}$$

$[N_R]$ is given by Eq. (A-3b) with

$$d_1 = d_2 = d_3 = 1 \text{ and } d_4 = -1 \quad \text{(A-4b)}$$

Case 3

$$[Y_R(y)]^T = \{\cosh n_1y\ ,\ \sinh n_1y\ ,\ \cosh n_2y\ ,\ \sinh n_2y,\ \cos n_3y\ ,\ \sin n_3y\ ,\ \cos n_4y\ ,\ \sin n_4y\} \tag{A-5a}$$

$[N_R]$ is given by Eq. (A-3b) with

$$d_1 = d_2 = 1\ ,\ \text{and}\ d_3 = d_4 = -1 \tag{A-5b}$$

Case 4

$$[Y_R(y)]^T = \{\cosh n_1y\ ,\ \sinh n_1y\ ,\ \cos n_2y\ ,\ \sin n_2y\ ,\ \cos n_3y\ ,\ \sin n_3y\ ,\ \cos n_4y\ ,\ \sin n_4y\} \tag{A-6a}$$

$[N_R]$ is given by Eq. (A-3b) with

$$d_1 = 1,\ \text{and}\ d_2 = d_3 = d_4 = -1 \tag{A-6b}$$

Case 5

$$[Y_R(y)]^T = \{\cos n_1y\ ,\ \sin n_1y\ ,\ \cos n_2y\ ,\ \sin n_2y\ \cos n_3y\ ,\ \sin n_3y\ ,\ \cos n_4y\ ,\ \sin n_4y\} \tag{A-7a}$$

$[N_R]$ is given by Eq. (A-3b) with

$$d_1 = d_2 = d_3 = d_4 = -1 \tag{A-7b}$$

Case 6

$$[Y_R(y)]^T = \{\cosh n_1y\ ,\ \sinh n_1y\ ,\ \cosh n_2y\ ,\ \sinh n_2y\ \cosh n_3y\ \cos n_4y\ ,\ \sinh n_3y\ \cos n_4y\ \sinh n_3y\ \sin n_4y\ ,\ \cosh n_3y\ \sin n_4y\} \tag{A-8a}$$

$$[N_R] = \begin{bmatrix} 0 & d_1 n_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ n_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2 n_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & n_3 & 0 & -n_4 \\ 0 & 0 & 0 & 0 & n_3 & 0 & -n_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & n_4 & 0 & n_3 \\ 0 & 0 & 0 & 0 & n_4 & 0 & n_3 & 0 \end{bmatrix} \qquad \text{(A-8b)}$$

where

$d_1 = d_2 = 1$

Case 7

$$[Y_R(y)]^T = \{\cosh n_1 y\ ,\ \sinh n_1 y\ ,\ \cos n_2 y\ ,\ \sin n_2 y\ ,$$
$$\cosh n_3 y\ ,\ \cos n_4 y\ ,\ \sinh n_3 y,\ \cos n_4 y\ ,$$
$$\sinh n_3 y\ ,\ \sin n_4 y\ ,\ \cosh n_3 y,\ \sin n_4 y\ \} \qquad \text{(A-9a)}$$

$[N_R]$ is given by Eq. (A-8b) with

$d = 1$, and $d_2 = -1$ (A.9b)

Case 8

$$[Y_R(y)]^T = \{\cos n_1 y\ ,\ \sin n_1 y\ ,\ \cos n_2 y\ ,\ \sin n_2 y,$$
$$\cosh n_3 y\ \cos n_4 y\ ,\ \sin n_3 y\ \ \cos n_4 y,$$
$$\sinh n_3 y\ \ \sin n_4 y\ ,\ \cosh n_3 y\ \ \sin n_4 y\} \qquad \text{(A-10a)}$$

$[N_R]$ is given by Eq. (A-8b) with

$d_1 = d_2 = -1$ (A-10b)

Case 9

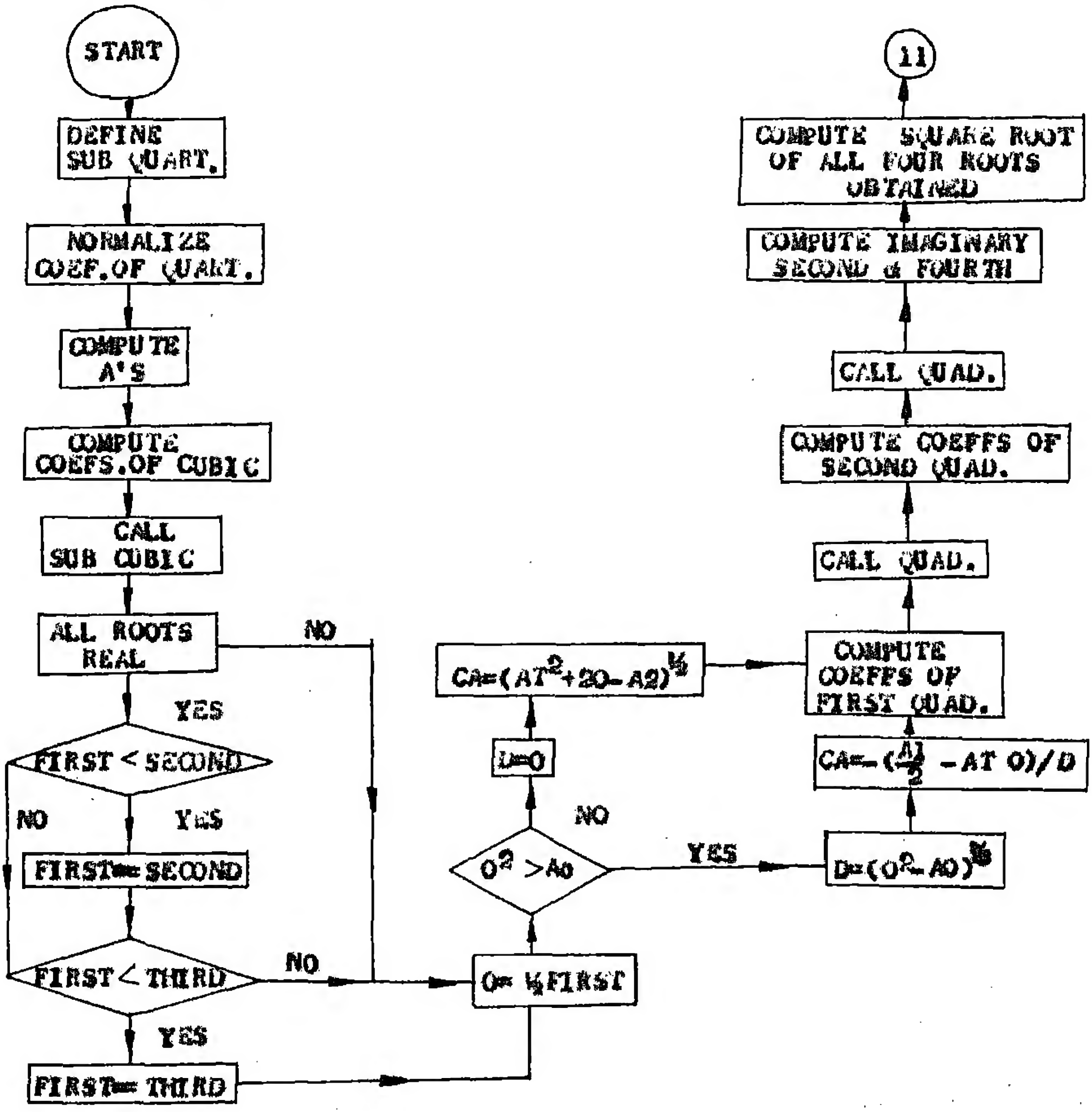
$$\left[Y_R(y)\right]^T = \left\{ \begin{array}{llll} \cosh a_1y & \cos a_2y, & \sinh a_1y & \cos a_2y, \\ \cosh a_1y & \sin a_2y, & \sinh a_1y & \sin a_2y, \\ \cosh a_3y & \cos a_4y, & \sinh a_3y & \cos a_4y, \\ \cosh a_3y & \sin a_4y, & \sinh a_3y & \sin a_4y \end{array} \right\} \tag{A-11a}$$

$$\left[N_R\right] = \begin{bmatrix} 0 & a_1 & 0 & -a_2 & 0 & 0 & 0 & 0 \\ a_1 & 0 & -a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & a_1 & 0 & 0 & 0 & 0 \\ a_2 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_3 & 0 & -a_4 \\ 0 & 0 & 0 & 0 & a_3 & 0 & -a_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_4 & 0 & a_3 \\ 0 & 0 & 0 & 0 & a_4 & 0 & a_3 & 0 \end{bmatrix} \tag{A-11b}$$

APPENDIX B

FLOW CHARTS

Main Flow Chart



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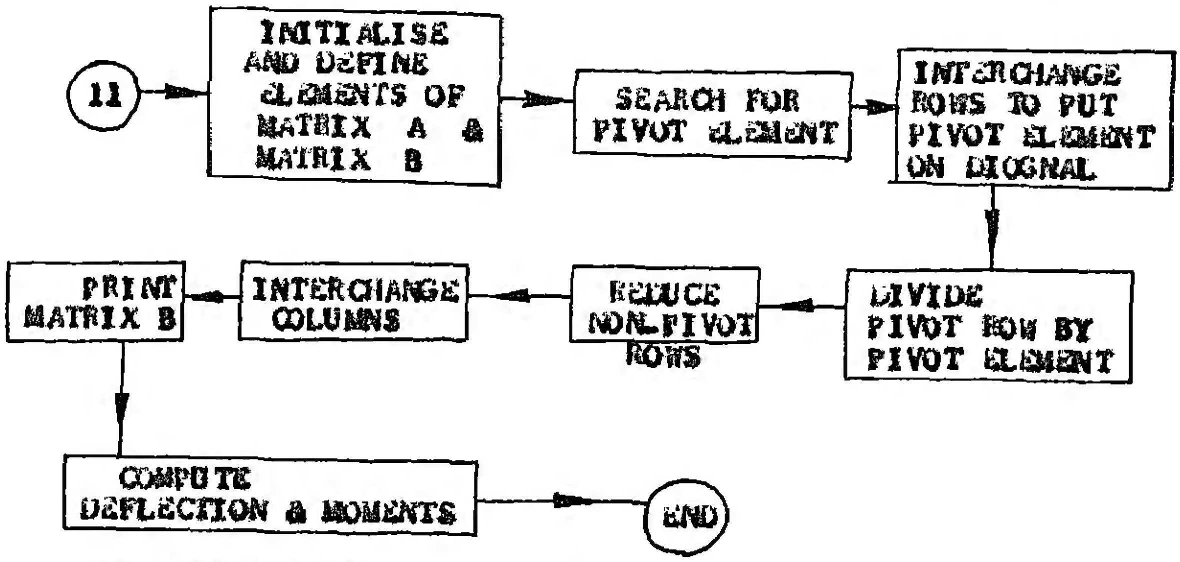
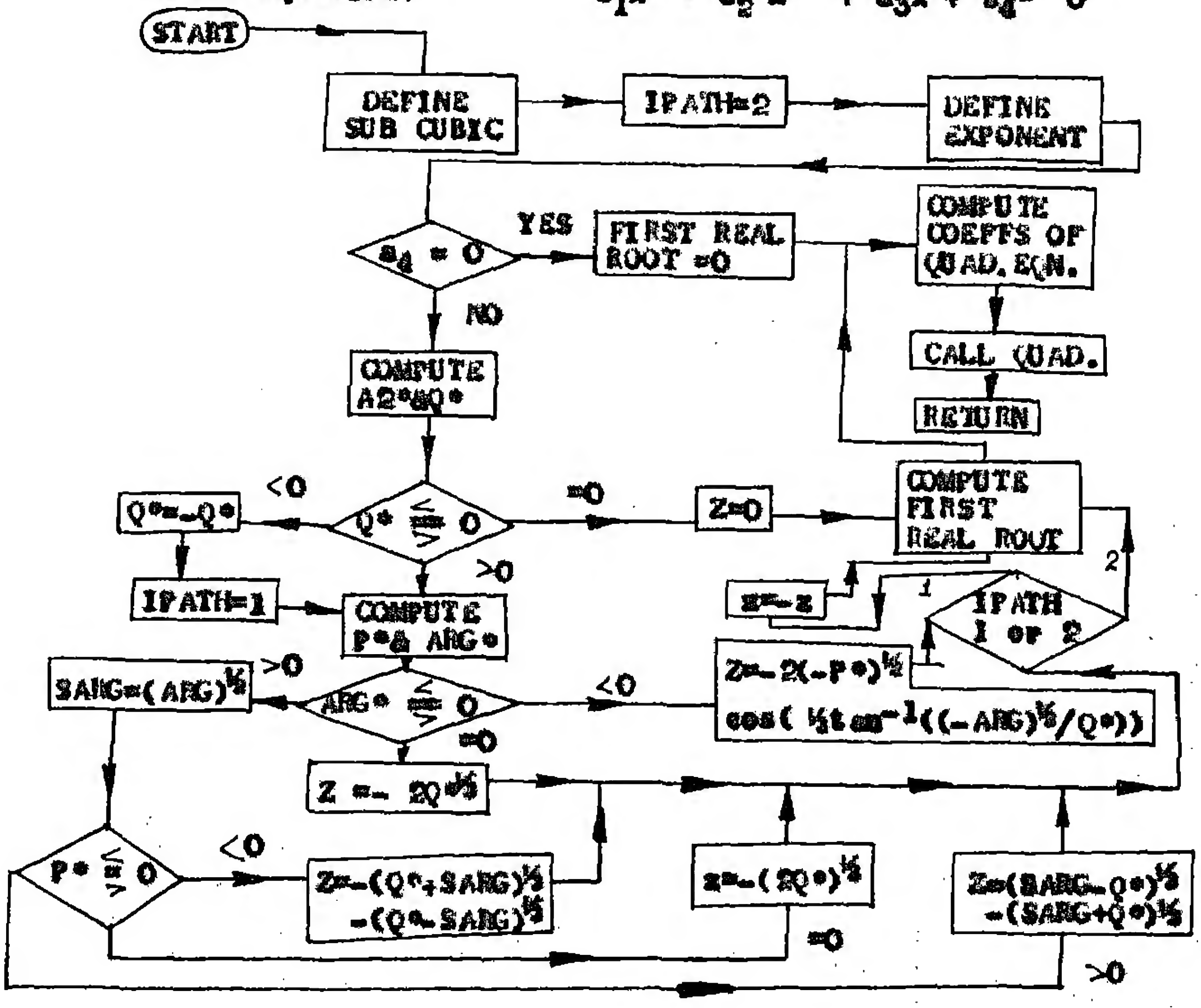
*Expressions are given in computer programme.

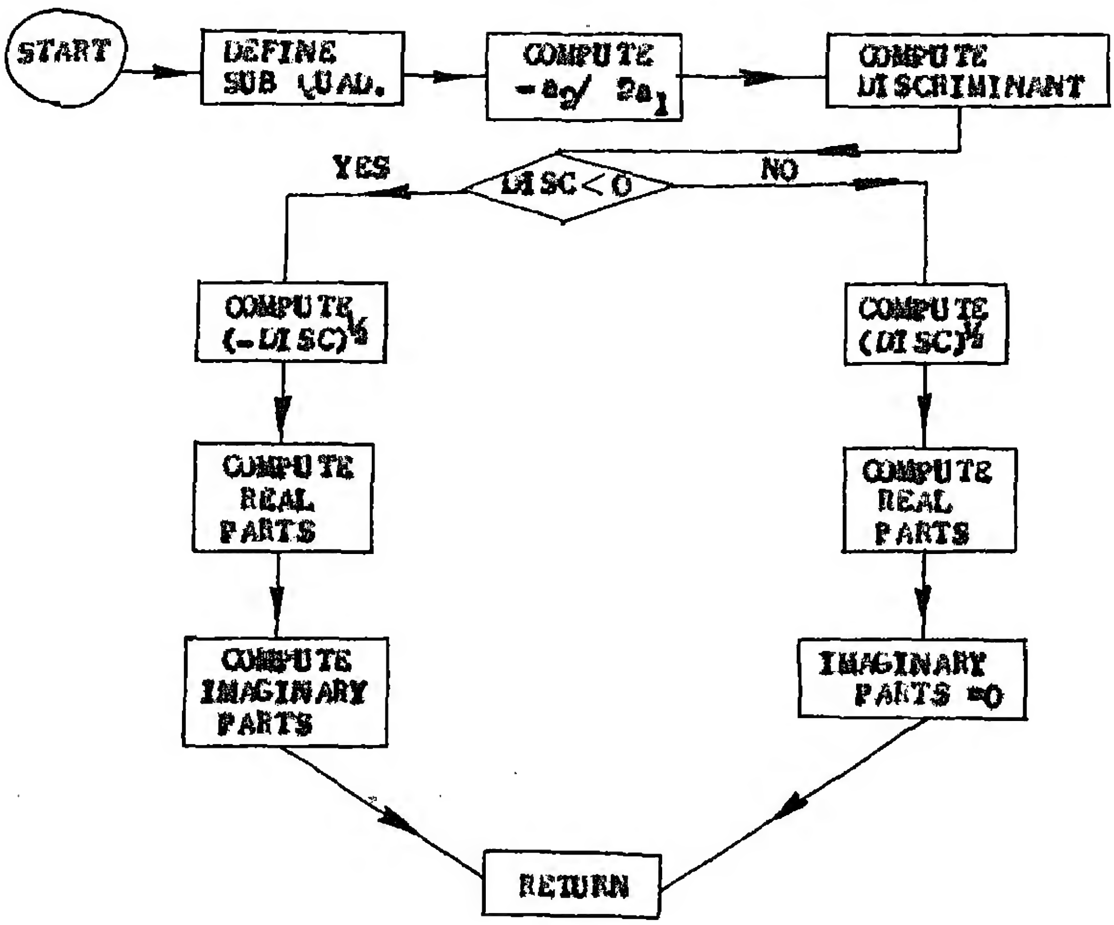


Sub Flow Charts

Cubic Equation: $a_1x^3 + a_2x^2 + a_3x + a_4 = 0$



Quadratic Equation: $a_1x^2 + a_2 x + a_3 = 0$



APPENDIX C

ALL EDGES SIMPLY SUPPORTED FORTRAN SOURCE LIST

SOURCE STATEMENT

```
$IBFTC         NODECK
      DIMENSION C(5),RT(4),AQ(4),BQ(4),XI(4),AC(4),XR(4)
      DIMENSION       THETA(4),RP(4),RR(4),FR(4,2),FI(4,2),PP(4)
      DIMENSION IPIVOT(8),A(8,8),B(8,1),INDEX(8,2),PIVOT(8)
      DIMENSION P(10),W(9),T(8),XM(5),Q(4),XYM(7)
C     R=B/A
      PM=1.0
      PRINT3
  3   FORMAT(1H1)
  4   R=0.5
      X=0.0
  2   F=1.0
  78  PII=3.1416
      FACTR=PII*PM
      V=0.3
      U=12.*(1.0-V**2)*X**2
      C(1)=1.0
      C(2)=-4.0*(FACTR )**2
      C(3)=6.0*(FACTR )**4+U
      C(4)=-2.*(FACTR )**2*((2.*FACTR)**4+U*R)
      C(5)=(FACTR )**4*((FACTR )**4+R**2*U)
      A3=C(2)/C(1)
      A2=C(3)/C(1)
      A1=C(4)/C(1)
      A0=C(5)/C(1)
      AT=A3/2.
      AC(1)=1.
      AC( 2)=-A2
      AC( 3)=A1*A3-4.*A0
      AC(4)=A0*(4.*A2-A3**2)-A1**2
      CALL CUBIC (AC,RT,RTI)
      IF(RTI)20,10,20
   10 IF(RT(1)-RT(2))11,12,12
   11 RT(1)=RT(2)
   12 IF(RT(1)-RT(3))13,20,20
   13 RT(1)=RT(3)
   20 Q=RT(1)/2.
      IF(Q**2-A0)22,22,24
   22 D=0.
      CA=SQRT (AT**2+2.*Q-A2)
       GO TO 25
   24 D=SQRT (Q**2-A0)
      CA=-(A1/2.-AT*Q)/D
   25 AQ(1)=1.
      AQ(2)=AT-CA
       AQ(3)=Q-D
      CALL QUAD(AQ,XR(1),XR(2),XI(1))
      BQ(1)=1.
      BQ(2)=AT+CA
      BQ(3)=Q+D
      CALL QUAD(BQ,XR(3),XR(4),XI(3))
      XI(2)=-XI(1)
      XI(4)=-XI(3)
      PRINT77,X
 77   FORMAT(1X,F7.2)
```

FORTRAN SOURCE LIST

SOURCE STATEMENT

```
      PRINT7,PP
 7    FORMAT(1X,F7.2)
      DO1 I=1,4
      PP(I)=XI(I)/XR(I)
      THETA(I)=ATAN(PP(I))
      PI=4.*ATAN(1.)
      RP(I)=SQRT(XR(I)**2+XI(I)**2)
      RR(I)=SQRT(RP(I))
      FR(I,1)=RR(I)*COS(PI+0.5*THETA(I))
      FI(I,2)=RR(I)*SIN(2.*PI+0.5*THETA(I))
 1    FR(I,2)=RR(I)*COS(2.*PI+0.5*THETA(I))
      PRINT8,((FR(I,J),I=1,4),J=1,2)
      PRINT8,((FI(I,J),I=1,4),J=1,2)
 8    FORMAT(1X,PE14.4)
C     INITIALIZATION
      Q(1)=FR(1,2)
      Q(2)=FR(2,2)
      Q(3)=FR(4,2)
      Q(4)=FI(4,2)
      G=3.1416
      P(1)=-R *(PM*G)**2+ Q(3)**2- Q(4)**2
      P(2)=2.*Q(3)*Q(4)
      P(3)=(PM*G)**4+Q(3)**4+Q(4)**4-6.*(Q(3)*Q(4))**2-(Q(3)**2-Q(4)**2)
     2*(PM*G)**2
      P(4)=4.*Q(3)**3*Q(4)-4.*Q(4)**3*Q(3)-2.*Q(4)*Q(3)*(PM*G)**2
      PK=P(3)**2+P(4)**2
      P(5)=(P(4)*P(1)-P(3)*P(2))/PK
      P(6)=(P(2)*P(4)+P(1)*P(3))/PK
      P(7)=(P(1)*P(3)+P(2)*P(4))/PK
      P(8)=(P(2)*P(3)-P(1)*P(4))/PK
      P(9)=(-R *(PM*G)**2+Q(1)**2)/((PM*G)**4+Q(1)**4-(PM*G*Q(1))**2)
      P(10)=(-R *(PM*G)**2+Q(2)**2)/((PM*G)**4+Q(2)**4-(PM*G*Q(2))**2)
      A(1,1)=1.0
      A(1,2)=0.0
      A(1,3)=1.0
      A(1,4)=0.0
      A(1,5)=1.0
      A(1,6)=0.0
      A(1,7)=0.0
      A(1,8)=0.0
      A(2,1)=COSH(Q(1)*F)
      A(2,2)=SINH (Q(1)*F)
      A(2,3)=COSH(Q(2)*F)
      A(2,4)=SINH (Q(2)*F)
      A(2,5)=COSH(Q(3)*F)*COS(Q(4)*F)
      A(2,6)=SINH (Q(3)*F)*COS(Q(4)*F)
      A(2,7)=COSH(Q(3)*F)*SIN(Q(4)*F)
      A(2,8)=SINH (Q(3)*F)*SIN(Q(4)*F)
      A(3,1)=Q(1)**2
      A(3,2)=0.0
      A(3,3)=Q(2)**2
      A(3,4)=0.0
      A(3,5)=Q(3)**2-Q(4)**2
      A(3,6)=0.0
      A(3,7)=0.0
```

FORTRAN SOURCE LIST

SOURCE STATEMENT

```
      A(3,8)=2.*C(3)*C(4)
      A(4,1)=C(1)**2*A(2,1)
      A(4,2)=C(1)**2*A(2,2)
      A(4,3)=C(2)**2*A(2,3)
      A(4,4)=C(2)**2*A(2,4)
      A(4,5)=A(3,5)*A(2,5)-P(2)*A(2,8)
      A(4,6)=A(3,5)*A(2,6)-P(2)*A(2,7)
      A(4,7)=(A(3,5)*A(2,7)+P(2)*A(2,6))
      A(4,8)=(A(3,5)*A(2,8)+P(2)*A(2,5))
      A(5,1)=P(9)
      A(5,2)=0.0
      A(5,3)=P(10)
      A(5,4)=0.0
      A(5,5)=P(7)
      A(5,6)=0.0
      A(5,7)=0.0
      A(5,8)=P(8)
      A(6,1)=P(9)*A(2,1)
      A(6,2)=P(9)*A(2,2)
      A(6,3)=P(10)*A(2,3)
      A(6,4)=P(10)*A(2,4)
      A(6,5)=P(7)*A(2,5)+P(5)*A(2,8)
      A(6,6)=P(7)*A(2,6)+P(5)*A(2,7)
      A(6,7)=P(8)*A(2,6)+P(6)*A(2,7)
      A(6,8)=P(8)*A(2,5)+P(6)*A(2,8)
      A(7,1)=C(1)**2*P(9)
      A(7,2)=0.0
      A(7,3)=C(2)**2*P(10)
      A(7,4)=0.0
      A(7,5)=P(7)*A(3,5)+P(2)*P(5)
      A(7,6)=0.0
      A(7,7)=0.0
      A(7,8)=P(8)*A(3,5)+P(2)*P(6)
      A(8,1)=A(4,1)*P(9)
      A(8,2)=A(4,2)*P(9)
      A(8,3)=A(4,3)*P(10)
      A(8,4)=A(4,4)*P(10)
      A(8,5)=P(7)*A(4,5)+P(5)*A(4,8)
      A(8,6)=P(7)*A(4,6)+P(5)*A(4,7)
      A(8,7)=P(8)*A(4,6)+P(6)*A(4,7)
      A(8,8)=P(8)*A(4,5)+P(6)*A(4,8)
      B(1,1)=-1.0
      B(2,1)=-1.0
      B(3,1)=-(PM*G)**2
      B(4,1)=-(PM*G)**2
      B(5,1)=0.0
      B(6,1)=0.0
      B(7,1)=0.0
      B(8,1)=0.0
      N=8
      M=1
      DETERM=0.0
   15 DO 21 J=1,N
   21 IPIVOT(J)=0
   30 DO 550 I=1,N
```

SOURCE STATEMENT

```
C
C      SEARCH FOR PIVOT ELEMENT
C
   40 AMAX=0.
   45 DO 105 J=1,N
   50 IF(IPIVOT(J)-1)60,105,60
   60 DO 100 K=1,N
   70 IF(IPIVOT(K)-1)80,100,740
   80 IF(ABS(AMAX)-ABS(A(J,K)))85,100,100
   85 IROW=J
   90 ICOLUM=K
   95 AMAX=A(J,K)
  100 CONTINUE
  105 CONTINUE
C
C      INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIDONAL
C
  110 IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1
  130 IF(IROW-ICOLUM)140,260,140
  140 DETERM=-DETERM
  150 DO 200 L=1,N
  160 SWAP=A(IROW,L)
  170 A(IROW,L)=A(ICOLUM,L)
  200 A(ICOLUM,L)=SWAP
  205 IF(M)260,260,210
  210 DO 250 L=1,M
  220 SWAP=B(IROW,L)
  230 B(IROW,L)=B(ICOLUM,L)
  250 B(ICOLUM,L)=SWAP
  260 INDEX(I,1)=IROW
  270 INDEX(I,2)=ICOLUM
  310 PIVOT(I)=A(ICOLUM,ICOLUM)
  320 DETERM=DETERM*PIVOT(I)
C
C      DIVIDE PIVOT ROW BY PIVOT ELEMENT
C
  330 A(ICOLUM,ICOLUM)=1.
  340 DO 350 L=1,N
  350 A(ICOLUM,L)=A(ICOLUM,L)/PIVOT(I)
  355 IF(M)380,380,360
  360 DO 370 L=1,M
  370 B(ICOLUM,L)=B(ICOLUM,L)/PIVOT(I)
C
C      REDUCE NON-PIVOT ROWS
C
  380 DO 550 L1=1,N
  390 IF(L1-ICOLUM)400,550,400
 400  TA=A(L1,ICOLUM)
  420 A(L1,ICOLUM)=0.
  430 DO 450 L=1,N
  450 A(L1,L)=A(L1,L)-A(ICOLUM,L)*TA
  455 IF(M)550,550,460
  460 DO 500 L=1,M
  500 B(L1,L)=B(L1,L)-B(ICOLUM,L)*TA
  550 CONTINUE
```

FORTRAN SOURCE LIST

SOURCE STATEMENT

```
C
C      INTERCHANGE COLUMNS
C
 600 DO 710 I=1,N
 610 L=N+1-I
 620 IF(INDEX(L,1)-INDEX(L,2))630,710,630
 630 JROW=INDEX(L,1)
 640 JCOLUM=INDEX(L,2)
 650 DO 705 K=1,N
 660 SWAP=A(K,JROW)
 670 A(K,JROW)=A(K,JCOLUM)
 700 A(K,JCOLUM)=SWAP
 705 CONTINUE
 710 CONTINUE
 740 CONTINUE
     PRINT820,(B(I,1),I=1,8)
 820  FORMAT(1X,8E15.4)
     Y=0.0
1900 T(1)=COSH(Q(3)*F*Y)
     T(2)=SINH (Q(3)*F*Y)
     T(3)=COS (Q(4)*F*Y)
     T(4)=SIN(Q(4)*F*Y)
     T(5)=COSH(Q(1)*F*Y)
     T(6)=SINH (Q(1)*F*Y)
     T(7)=COSH(Q(2)*F*Y)
     T(8)=SINH (Q(2)*F*Y)
     W(1)=B(1,1)*T(5)
     W(2)=B(2,1)*T(6)
     W(3)=B(3,1)*T(7)
     W(4)=B(4,1)*T(8)
     W(5)=B(5,1)*T(1)*T(3)
     W(6)=B(6,1)*T(2)*T(3)
     W(7)=B(7,1)*T(1)*T(4)
     W(8)=B(8,1)*T(2)*T(4)
     W(9)=W(1)+W(2)+W(3)+W(4)+W(5)+W(6)+W(7)+W(8)+1.0
     WW=5.2*W(9)/(PM*G)**3
     XM(1)=-W(9)
     XM(2)=Q(1)**2*(B(1,1)*T(5)+B(2,1)*T(6))
     XM(3)=Q(2)**2*(B(3,1)*T(7)+B(4,1)*T(8))
     WK=W(5)+W(6)+W(7)+W(8)
     XM(4)=(Q(3)**2-Q(4)**2)*WK
     XM(5)=2.*Q(3)*Q(4)*(-B(5,1)*T(2)*T(4)-B(6,1)*T(1)*T(4)+B(7,1)*T(2)
    2*T(3)+B(8,1)*T(1)*T(3))
     ZM=(XM(2)+XM(3)+XM(4)+XM(5))/(PM*G)**2
     XMM=-(XM(1)+.3*ZM+1.0)/(2.1*PM*G)
     YMM=-(ZM+.3*XM(1)+1.)/(2.1*PM*G)
     XYM(1)=Q(1)*(B(1,1)*T(6)+B(2,1)*T(5))
     XYM(2)=Q(2)*(B(3,1)*T(8)+B(4,1)*T(7))
     XYM(3)=Q(3)*(B(5,1)*T(2)*T(3)+B(6,1)*T(1)*T(3))
     XYM(4)=Q(3)*( B(7,1)*T(2)*T(4)+B(8,1)*T(1)*T(4))
     XYM(5)=Q(4)*(-B(5,1)*T(1)*T(4)-B(6,1)*T(2)*T(4))
     XYM(5)=Q(4)*(-B(5,1)*T(1)*T(4)-B(6,1)*T(2)*T(4))
     XYM(6)=Q(4)*(B(7,1)*T(1)*T(3)+B(8,1)*T(2)*T(3))
     XYM(7)=XYM(1)+XYM(2)+XYM(3)+XYM(4)+XYM(5)+XYM(6)
     XKM=-XYM(7)/(3.0*(PM*G)**2)
```

FORTRAN SOURCE LIST

```
        SOURCE STATEMENT

        PRINT2000,Y,WW,XMM,YMM,XKM
  2000  FORMAT(1X,5E15.4)
        Y=Y+.1
        IF(Y-1.0)1900,1900,2100
  2100  F=F+.2
        IF(F-2.)78,78,2200
  2200  X=X+5.0
        IF(X-50.0 )2,2,6
  6     PM=PM+2.0
        IF(PM-7.)4,4,9
  9     STOP
        END
```

IBMAP ASSEMBLY

ESSAGES FOR ABOVE ASSEMBLY

FORTRAN SOURCE LIST

```
N        SOURCE STATEMENT

 $IBFTC CUBIC    NODECK
        SUBROUTINE CUBIC(A,XR,XI)
        DIMENSION A(4),XR(3),AQ(3)
         IPATH=2
        EX=1./3.
  1050  FORMAT(5F14.2)
        IF(A(4))1006,1004,1006
  1004  XR(1)=0.
        GO TO 1034
  1006  A2=A(1)**2
        Q=(27.*A2*A(4)-9.*A(1)*A(2)*A(3)+2.*A(2)**3)/(54.*A2*A(1))
        IF(Q)1010,1008,1014
  1008  Z=0.
        GO TO 1032
  1010  Q=-Q
        IPATH=1
  1014  P=(3.*A(1)*A(3)-A(2)**2)/(9.*A2)
        ARG=P**3+Q**2
        IF(ARG)1016,1018,1020
  1016  Z=-2.*SQRT(-P)*COS(ATAN(SQRT(-ARG)/Q)/3.)
        GO TO 1028
  1018  Z=-2.*Q**EX
        GO TO 1028
  1020  SARG=SQRT(ARG)
        IF(P)1022,1024,1026
  1022  Z=-(Q+SARG)**EX-(Q-SARG)**EX
        GO TO 1028
  1024  Z=-(2.*Q)**EX
        GO TO 1028
  1026  Z=(SARG-Q)**EX-(SARG+Q)**EX
  1028  GO TO(1030,1032),IPATH
  1030  Z=-Z
  1032  XR(1)=(3.*A(1)*Z-A(2))/(3.*A( 1))
  1034  AQ(1)=A(1)
        AQ(2)=A(2)+XR(1)*A(1)
        AQ(3)=A(3)+XR(1)*AQ(2)
        CALL QUAD(AQ,XR(2),XR(3),XI)
        RETURN
        END
```

IBMAP ASSEMBLY CUBIC

ESSAGES FOR ABOVE ASSEMBLY

FORTRAN SOURCE LIST

```
N       SOURCE STATEMENT

  $IBFTC QUAD     NODECK
        SUBROUTINE QUAD(A,XR1,XR2,XI)
        DIMENSION A(3)
     26 FORMAT(5F14.3)
        X1=-A(2)/(2.*A(1))
         DISC=X1**2-A(3)/A(1)
        IF(DISC)27,28,28
     27 X2=SQRT(-DISC)
         XR1=X1
         XR2=X1
         XI=X2
        GO TO 29
     28 X2=SQRT(DISC)
        XR1=X1+X2
         XR2=X1-X2
        XI=0.
     29 RETURN
        END
```

IBMAP ASSEMBLY QUAD

ESSAGES FOR ABOVE ASSEMBLY

IBLDR -- JOB   000000

IS BEING ENTERED INTO STORAGE.

FORTRAN SOURCE LIST

```
      SOURCE STATEMENT

 $IBFTC          NODECK
      DIMENSION C(5),RT(4),AQ(4),BQ(4),XI(4),AC(4),XR(4)
      DIMENSION      THETA(4),RP(4),PR(4),FR(4,2),FI(4,2),PP(4)
      DIMENSION IPIVOT(8),A(5,2),B(5,1),INDEX(5,2),PIVOT(8)
      DIMENSION P(10),W(9),T(3),XM(5),Q(4),XYM(7)
C     F=B/A
      PM=1.0
      PRINT3
    3 FORMAT(1H1)
    4 R=0.5
      X=0.0
    2 F=1.0
   76 PII=3.1416
      FACTR=PII*PM
      V=0.3
      U=12.*(1.0-V**2)*X**2
      C(1)=1.0
      C(2)=-4.0*(FACTR  )**2
      C(3)=6.0*(FACTR  )**4+U
      C(4)=-2.*(FACTR  )**2*((2.*FACTR)**4+U*R)
      C(5)=(FACTR  )**4*((FACTR  )**4+R**2*U)
      A3=C(2)/C(1)
      A2=C(3)/C(1)
      A1=C(4)/C(1)
      A0=C(5)/C(1)
      AT=A3/2.
      AC(1)=1.
      AC( 2)=-A2
      AC( 3)=A1*A3-4.*A0
      AC(4)=A0*(4.*A2-A3**2)-A1**2
      CALL CUBIC (AC,RT,RTI)
      IF(RTI)20,10,20
   10 IF(RT(1)-RT(2))11,12,12
   11 RT(1)=RT(2)
   12 IF(RT(1)-RT(3))13,20,20
   13 RT(1)=RT(3)
   20 O=RT(1)/2.
      IF(O**2-A0)22,22,24
   22 D=0.
      CA=SQRT (AT**2+2.*O-A2)
      GO TO 25
   24 D=SQRT (O**2-A0)
      CA=-(A1/2.-AT*O)/D
   25 AQ(1)=1.
      AQ(2)=AT-CA
      AQ(3)=O-D
      CALL QUAD(AQ,XR(1),XR(2),XI(1))
      BQ(1)=1.
      BQ(2)=AT+CA
      BQ(3)=O+D
      CALL QUAD(BQ,XR(3),XR(4),XI(3))
      XI(2)=-XI(1)
      XI(4)=-XI(3)
      PRINT77,X
   77 FORMAT(1X,F7.2)
```

FORTRAN SOURCE LIST

SOURCE STATEMENT

```
      PRINT7,PM
    7 FORMAT(1X,F7.2)
      DO1 I=1,4
      PP(I)=XI(I)/XR(I)
      THETA(I)=ATAN(PP(I))
      PI=4.*ATAN(1.)
      RP(I)=SQRT(XR(I)**2+XI(I)**2)
      RR(I)=SQRT(RP(I))
      FR(I,1)=RR(I)*COS(PI+0.5*THETA(I))
      FI(I,1)=RR(I)*SIN(PI+0.5*THETA(I))
      FI(I,2)=RR(I)*SIN(2.*PI+0.5*THETA(I))
    1 FR(I,2)=RR(I)*COS(2.*PI+0.5*THETA(I))
      PRINT8,((FR(I,J),I=1,4),J=1,2)
      PRINT8,((FI(I,J),I=1,4),J=1,2)
    8 FORMAT(1X,8E14.4)
C     INITIALIZATION
      Q(1)=FR(1,2)
      Q(2)=FR(2,2)
      Q(3)=FR(4,2)
      Q(4)=FI(4,2)
      G=3.1416
      P(1)=-.5*(PM*G)**2+ Q(3)**2- Q(4)**2
      P(2)=2.*Q(3)*Q(4)
      P(3)=(PM*G)**4+Q(3)**4+Q(4)**4-6.*(Q(3)*Q(4))**2-(Q(3)**2-Q(4)**2)
     2*(PM*G)**2
      P(4)=4.*Q(3)**3*Q(4)-4.*Q(4)**3*Q(3)-2.*Q(4)*Q(3)*(PM*G)**2
      PK=P(3)**2+P(4)**2
      P(5)=(P(4)*P(1)-P(3)*P(2))/PK
      P(6)=(P(2)*P(4)+P(1)*P(3))/PK
      P(7)=(P(1)*P(3)+P(2)*P(4))/PK
      P(8)=(P(2)*P(3)-P(1)*P(4))/PK
      P(9)=(-.5*(PM*G)**2+Q(1)**2)/((PM*G)**4+Q(1)**4-(PM*G*Q(1))**2)
      P(10)=(-.5*(PM*G)**2+Q(2)**2)/((PM*G)**4+Q(2)**4-(PM*G*Q(2))**2)
      A(1,1)=1.0
      A(1,2)=0.0
      A(1,3)=1.0
      A(1,4)=0.0
      A(1,5)=1.0
      A(1,6)=0.0
      A(1,7)=0.0
      A(1,8)=0.0
      A(2,1)=COSH(Q(1)*F)
      A(2,2)=SINH (Q(1)*F)
      A(2,3)=COSH(Q(2)*F)
      A(2,4)=SINH (Q(2)*F)
      A(2,5)=COSH(Q(3)*F)*COS(Q(4)*F)
      A(2,6)=SINH (Q(3)*F)*COS(Q(4)*F)
      A(2,7)=COSH(Q(3)*F)*SIN(Q(4)*F)
      A(2,8)=SINH (Q(3)*F)*SIN(Q(4)*F)
      A(3,1)=0.0
      A(3,2)=Q(1)
      A(3,3)=0.0
      A(3,4)=Q(2)
      A(3,5)=0.0
      A(3,6)=Q(3)
```

```
        SOURCE STATEMENT

      A(3,7)=Q(4)
      A(3,8)=0.0
      A(4,1)=Q(1)*A(2,2)
      A(4,2)=Q(1)*A(2,1)
      A(4,3)=Q(2)*A(2,4)
      A(4,4)=Q(2)*A(2,3)
      A(4,5)=Q(3)*A(2,6)-Q(4)*A(2,7)
      A(4,6)=Q(3)*A(2,5)-Q(4)*A(2,8)
      A(4,7)=Q(3)*A(2,8)+Q(4)*A(2,5)
      A(4,8)=Q(3)*A(2,7)+Q(4)*A(2,6)
      A(5,1)=P(9)
      A(5,2)=0.0
      A(5,3)=P(10)
      A(5,4)=0.0
      A(5,5)=P(7)
      A(5,6)=0.0
      A(5,7)=0.0
      A(5,8)=P(8)
      A(6,1)=P(9)*A(2,1)
      A(6,2)=P(9)*A(2,2)
      A(6,3)=P(10)*A(2,3)
      A(6,4)=P(10)*A(2,4)
      A(6,5)=P(7)*A(2,5)+P(5)*A(2,8)
      A(6,6)=P(7)*A(2,6)+P(5)*A(2,7)
      A(6,7)=P(8)*A(2,6)+P(6)*A(2,7)
      A(6,8)=P(8)*A(2,5)+P(6)*A(2,8)
      P(11)=Q(3)**2-Q(4)**2
      P(12)=Q(1)**2*A(2,1)
      P(13)=Q(1)**2*A(2,2)
      P(14)=Q(2)**2*A(2,3)
      P(15)=Q(2)**2*A(2,4)
      P(16)=P(11)*A(2,5)-P(2)*A(2,8)
      P(17)=P(11)*A(2,6)-P(2)*A(2,7)
      P(18)=P(11)*A(2,7)+P(2)*A(2,6)
      P(19)=P(11)*A(2,8)+P(2)*A(2,5)
      A(7,1)=Q(1)**2*P(9)
      A(7,2)=0.0
      A(7,3)=Q(2)**2*P(10)
      A(7,4)=0.0
      A(7,5)=P(7)*P(11) +P(2)*P(5)
      A(7,6)=0.0
      A(7,7)=0.0
      A(7,8)=P(8)*P(11) +P(2)*P(6)
      A(8,1)=P(12) *P(9)
      A(8,2)=P(13) *P(9)
      A(8,3)=P(14) *P(10)
      A(8,4)=P(15) *P(10)
      A(8,5)=P(7)*P(16) +P(5)*P(19)
      A(8,6)=P(7)*P(17) +P(5)*P(18)
      A(8,7)=P(8)*P(17) +P(6)*P(18)
      A(8,8)=P(8)*P(16) +P(6)*P(19)
      B(1,1)=-1.0
      B(2,1)=-1.0
      B(3,1)=0.0
      B(4,1)=0.0
```

```
        SOURCE STATEMENT

        B(5,1)=0.0
        B(6,1)=0.0
        B(7,1)=0.0
        B(8,1)=0.0
        N=8
        M=1
        DETERM=0.0
     19 DO 21 J=1,N
  21    IPIVOT(J)=0
     30 DO 550 I=1,N
C
C       SEARCH FOR PIVOT ELEMENT
C
     40 AMAX=0.
     45 DO 105 J=1,N
     50 IF(IPIVOT(J)-1)60,105,60
     60 DO 100 K=1,N
     70 IF(IPIVOT(K)-1)80,100,740
     80 IF(ABS(AMAX)-ABS(A(J,K)))85,100,100
     85 IROW=J
     90 ICOLUM=K
     95 AMAX=A(J,K)
    100 CONTINUE
    105 CONTINUE
C
C       INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIOGNAL
C
    110 IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1
    130 IF(IROW-ICOLUM)140,260,140
    140 DETERM=-DETERM
    150 DO 200 L=1,N
    160 SWAP=A(IROW,L)
    170 A(IROW,L)=A(ICOLUM,L)
    200 A(ICOLUM,L)=SWAP
    205 IF(M)260,260,210
    210 DO 250 L=1,M
    220 SWAP=B(IROW,L)
    230 B(IROW,L)=B(ICOLUM,L)
    250 B(ICOLUM,L)=SWAP
    260 INDEX(I,1)=IROW
    270 INDEX(I,2)=ICOLUM
    310 PIVOT(I)=A(ICOLUM,ICOLUM)
    320 DETERM=DETERM*PIVOT(I)
C
C       DIVIDE PIVOT ROW BY PIVOT ELEMENT
C
    330 A(ICOLUM,ICOLUM)=1.
    340 DO 350 L=1,N
    350 A(ICOLUM,L)=A(ICOLUM,L)/PIVOT(I)
    355 IF(M)380,380,360
    360 DO 370 L=1,M
    370 B(ICOLUM,L)=B(ICOLUM,L)/PIVOT(I)
C
C       REDUCE NON-PIVOT ROWS
C
```

ISN SOURCE STATEMENT

```
  380 DO 550 L1=1,N
  390 IF(L1-ICOLUM)400,550,400
  400 TA=A(L1,ICOLUM)
  420 A(L1,ICOLUM)=0.
  430 DO 450 L=1,N
  450 A(L1,L)=A(L1,L)-A(ICOLUM,L)*TA
  455 IF(M)550,550,460
  460 DO 500 L=1,M
  500 B(L1,L)=B(L1,L)-B(ICOLUM,L)*TA
  550 CONTINUE
C
C       INTERCHANGE COLUMNS
C
  600 DO 710 I=1,N
  610 L=N+1-I
  620 IF(INDEX(L,1)-INDEX(L,2))630,710,630
  630 JROW=INDEX(L,1)
  640 JCOLUM=INDEX(L,2)
  650 DO 705 K=1,N
  660 SWAP=A(K,JROW)
  670 A(K,JROW)=A(K,JCOLUM)
  700 A(K,JCOLUM)=SWAP
  705 CONTINUE
  710 CONTINUE
  740 CONTINUE
      PRINT820,(B(I,1),I=1,8)
 820  FORMAT(1X,8E15.4)
      Y=0.0
 1900 T(1)=COSH(Q(3)*F*Y)
      T(2)=SINH (Q(3)*F*Y)
      T(3)=COS (Q(4)*F*Y)
      T(4)=SIN(Q(4)*F*Y)
      T(5)=COSH(Q(1)*F*Y)
      T(6)=SINH (Q(1)*F*Y)
      T(7)=COSH(Q(2)*F*Y)
      T(8)=SINH (Q(2)*F*Y)
      W(1)=B(1,1)*T(5)
      W(2)=B(2,1)*T(6)
      W(3)=B(3,1)*T(7)
      W(4)=B(4,1)*T(8)
      W(5)=B(5,1)*T(1)*T(3)
      W(6)=B(6,1)*T(2)*T(3)
      W(7)=B(7,1)*T(1)*T(4)
      W(8)=B(8,1)*T(2)*T(4)
      W(9)=W(1)+W(2)+W(3)+W(4)+W(5)+W(6)+W(7)+W(8)+1.0
      WM=5.2*W(9)/(PM*G)**3
      XM(1)=-W(9)
      XM(2)=Q(1)**2*(B(1,1)*T(5)+B(2,1)*T(6))
      XM(3)=Q(2)**2*(B(3,1)*T(7)+B(4,1)*T(8))
      WK=W(5)+W(6)+W(7)+W(8)
      XM(4)=(Q(3)**2-Q(4)**2)*WK
      XM(5)=2.*Q(3)*Q(4)*(-B(5,1)*T(2)*T(4)-B(6,1)*T(1)*T(4)+B(7,1)*
     2*T(3)+B(8,1)*T(1)*T(3))
      ZM=(XM(2)+XM(3)+XM(4)+XM(5))/(PM*G)**2
      XMM=-(XM(1)+.3*ZM+1.0)/(2.1*PM*G)
```

FORTRAN SOURCE LIST

```
        SOURCE STATEMENT

        YMM=-(ZM+.3*XM(1)+1.)/(2.1*PM*G)
        XYM(1)=G(1)*(B(1,1)*T(6)+B(2,1)*T(5))
        XYM(2)=G(2)*(B(3,1)*T(8)+B(4,1)*T(7))
        XYM(3)=G(3)*(B(5,1)*T(2)*T(3)+B(6,1)*T(1)*T(3))
        XYM(4)=G(3)*( B(7,1)*T(2)*T(4)+B(8,1)*T(1)*T(4))
        XYM(5)=G(4)*(-B(5,1)*T(1)*T(4)-B(6,1)*T(2)*T(4))
        XYM(5)=G(4)*(-B(5,1)*T(1)*T(4)-B(6,1)*T(2)*T(4))
        XYM(6)=G(4)*(B(7,1)*T(1)*T(3)+B(8,1)*T(2)*T(3))
        XYM(7)=XYM(1)+XYM(2)+XYM(3)+XYM(4)+XYM(5)+XYM(6)
        XKM=-XYM(7)/(4.0*(PM*G)**2)
        PRINT2000,Y,WW,XMM,YMM,XKM
   2000 FORMAT(1X,5E15.4)
        Y=Y+.1
        IF(Y-1.0)1900,1900,2100
   2100 F=F+.2
        IF(F-2.)78,78,2200
   2200 X=X+5.0
        IF(X-50.0 )2,2,6
   6    PM=PM+2.0
        IF(PM-7.)4,4,9
   9    STOP
        END
```

IBMAP ASSEMBLY

MESSAGES FOR ABOVE ASSEMBLY

FORTRAN SOURCE LIST

```
          SOURCE STATEMENT

 $IBFTC CUBIC     NODECK
          SUBROUTINE CUBIC(A,XR,XI)
          DIMENSION A(4),XR(3),AQ(3)
           IPATH=2
          EX=1./3.
    1050 FORMAT(5F14.2)
          IF(A(4))1006,1004,1006
    1004 XR(1)=0.
          GO TO 1034
    1006 A2=A(1)**2
          Q=(27.*A2*A(4)-9.*A(1)*A(2)*A(3)+2.*A(2)**3)/(54.*A2*A(1))
          IF(Q)1010,1008,1014
    1008 Z=0.
          GO TO 1032
    1010 Q=-Q
          IPATH=1
    1014 P=(3.*A(1)*A(3)-A(2)**2)/(9.*A2)
          ARG=P**3+Q**2
          IF(ARG)1016,1018,1020
    1016 Z=-2.*SQRT(-P)*COS(ATAN(SQRT(-ARG)/Q)/3.)
          GO TO 1028
    1018 Z=-2.*Q**EX
          GO TO 1028
    1020 SARG=SQRT(ARG)
          IF(P)1022,1024,1026
    1022 Z=-(Q+SARG)**EX-(Q-SARG)**EX
          GO TO 1028
    1024 Z=-(2.*Q)**EX
          GO TO 1028
    1026 Z=(SARG-Q)**EX-(SARG+Q)**EX
    1028 GO TO(1030,1032),IPATH
    1030 Z=-Z
    1032 XR(1)=(3.*A(1)*Z-A(2))/(3.*A( 1))
    1034 AQ(1)=A(1)
          AQ(2)=A(2)+XR(1)*A(1)
          AQ(3)=A(3)+XR(1)*AQ(2)
          CALL QUAD(AQ,XR(2),XR(3),XI)
          RETURN
          END
```

IBMAP ASSEMBLY CUBIC

MESSAGES FOR ABOVE ASSEMBLY

```
                                        FORTRAN SOURCE LIST
          SOURCE STATEMENT

   $IBFTC QUAD     NODECK
          SUBROUTINE QUAD(A,XR1,XR2,XI)
          DIMENSION A(3)
       26 FORMAT(5F14.3)
          X1=-A(2)/(2.*A(1))
           DISC=X1**2-A(3)/A(1)
          IF(DISC)27,28,28
       27 X2=SQRT(-DISC)
           XR1=X1
           XR2=X1
           XI=X2
          GO TO 29
       28 X2=SQRT(DISC)
          XR1=X1+X2
           XR2=X1-X2
          XI=0.
       29 RETURN
          END
                                        IBMAP ASSEMBLY QUAD


MESSAGES FOR ABOVE ASSEMBLY
                                        IBLDR -- JOB    000000


M IS BEING ENTERED INTO STORAGE.
```

APPENDIX 'D'

Tabulation of Deflection and Moments for different Values of Parameter a^2/hR_1 and b/a ratio

$a^2/hR_1 = 0.5$, $b/a = 1.0$ $x/a = 0.5$ $R_1/R_2 = 0.5$

Y/b	$10^3 wh/\lambda Ta^2$	$10^2 M_x/E\lambda Th^2$	$10^2 M_y/E\lambda Th^2$	$10^3 M_{xy}/E\lambda Th^2$
0.0	00.00	-10.61	0.0	-47.61
0.1	27.33	-18.97	-20.36	-55.38
0.2	51.16	-22.42	-25.34	-39.04
0.3	65.91	-23.25	-24.07	-21.01
0.4	72.95	-23.11	-21.65	- 8.33
0.5	74.92	-22.95	-20.60	0.00
0.6	72.96	-23.11	-21.65	8.33
0.7	65.91	-23.25	-24.07	21.01
0.8	51.16	-22.42	-25.34	39.04
0.9	27.33	-18.97	-20.36	55.39
1.0	0.00	-10.61	0.00	47.61

$a^2/hR_1 = 0.5$, $b/a = 1.2$ $x/a = 0.5$ $R_1/R_2 = 0.5$

Y/b	$10^3 wh/\lambda Ta^2$	$10^2 M_x/E\lambda Th^2$	$10^2 M_y/E\lambda Th^2$	$10^3 M_{xy}/E\lambda Th^2$
0.0	0.00	-10.61	000.00	-38.48
0.1	27.23	-19.52	-22.24	-43.63
0.2	46.94	-21.93	-24.85	-21.71
0.3	54.16	-21.37	-21.02	-04.33
0.4	54.44	-20.20	-17.04	1.62
0.5	53.75	-19.68	-15.48	0.00
0.6	54.44	-20.20	-17.40	- 1.62
0.7	54.16	-21.37	-21.02	4.33
0.8	46.94	-21.93	-24.85	21.71
0.9	27.23	-19.53	-22.24	43.63
1.0	00.00	-10.61	00.00	38.48

a^2/hR_1=0 8.0 b/a=104 x/1=0.5 R_1/R_2=0.5

Y/b	$10^3wh/\perp Ta^2$	$10^2Mx/E\perp Th^2$	$10^2My/E\perp Th^2$	$10^2Mxy/E\perp Th^2$
0.0	0.0	-10.61	0.00	-34.73
0.1	28.63	-20.18	-23.96	-36.06
0.2	44.02	-21.55	-24.41	- 8.35
0.3	43.06	-19.83	-18.91	8.26
0.4	36.34	-17.89	-14.32	8.80
0.5	35.00	-17.13	-12.67	- 8.81
0.6	36.34	-17.93	-14.32	- 8.26
0.7	43.07	-19.83	-18.91	8.35
0.8	44.03	-21.55	-24.41	36.07
0.9	28.64	-20.18	-23.96	34.75
1.0	0.00	-10.61	0.00	34 .75

a^2/hR_1=8.0 b/a=1.6 x/a=0.5 R_1/R_2=0.5

Y/b	$10^3wh/\perp Ta^2$	$10^2Mx/E\perp Th^2$	$10^2My/E\perp Th^2$	$10^2Mxy/E\perp Th^2$
0.0	0.0	-10.61	0.00	-33.33
0.1	30.76	-20.70	-25.19	-30.34
0.2	41.54	-21.10	-23.57	+ 1.88
0.3	32.76	-18.47	-17.23	16.89
0.4	20.12	-16.13	-12.89	12.88
0.5	14.72	-15.27	-11.48	0.00
0.6	20.12	-15.13	-12.89	-12.88
0.7	32.76	-18.47	-17.23	-16.89
0.8	41.54	-21.10	-23.57	- 1.88
0.9	30.76	-20.70	-25.19	30.36
1.0	0.00	-10.61	0.00	33.40

a^2/hR_1=8.0 b/a=1.8 x/a=0.5 R_1/R_2=0.5

Y/b	$10^3wh/\perp Ta^2$	$10^2Mx/E\perp Th^2$	$10^2My/E\perp Th^2$	$10^2Mxy/E\perp Th^2$
0.0	0.00	-10.61	0.00	-32.48
0.1	32.73	-21.07	-25.90	-25.06
0.2	38.37	-20.46	-22.32	9.72
0.3	22.78	-17.22	-15.80	21.10
0.4	6.13	-14.80	-12.28	14.14
0.5	- 0.62	-13.97	-11.33	0.00
0.6	6.13	-14.80	-12.28	-14.14
0.7	22.78	-17.23	-15.80	-21.11
0.8	38.39	-20.46	-22.32	- 9.72
0.9	32.74	-21.08	-25.91	25.06
1.0	0.00	-10.62	0.00	32.65

$a^2/hR_1=5.0$ $b/a=2.0$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3Wh/\lambda Ta^2$	$10^2Mx/E\lambda Th^2$	$10^2My/E\lambda Th^2$	$10^2Mxy/E\lambda Th^2$
0.0	0.00	-10.61	0.00	-31.56
0.1	34.08	-21.27	-26.18	-19.71
0.2	33.97	-19.65	-20.82	15.71
0.3	12.84	-16.05	-14.60	22.69
0.4	- 5.76	-13.78	-12.15	13.38
0.5	-12.51	-13.09	-11.70	0.00
0.6	- 5.77	-13.78	-12.15	-13.38
0.7	12.82	-16.04	-14.60	-22.68
0.8	33.95	-19.65	-20.82	-15.67
0.9	34.08	-21.27	-26.20	19.94
1.0	0.00	-10.64	00.00	32.88

$a^2/hR_1=10.0$ $b/a=1.0$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3Wh/\lambda Ta^2$	$10^2Mx/E\lambda Th^2$	$10^2My/E\lambda Th^2$	$10^2Mxy/E\lambda Th^2$
0.0	0.00	-10.61	0.00	-20.96
0.1	14.37	-17.35	-18.54	-30.38
0.2	26.80	-19.45	-22.13	-18.55
0.3	32.89	-19.34	-20.08	- 6.81
0.4	34.62	-18.67	-17.36	- 1.15
0.5	34.80	-18.35	-16.24	0.00
0.6	34.62	-18.67	-17.36	1.15
0.7	32.89	-19.34	-20.08	6.81
0.8	26.80	-19.45	-22.13	18 .55
0.9	14.37	-17.35	-18.54	30.38
1.0	0.00	-10.61	0.00	20.96

$a^2/hR_1=10.0$ $b/a=1.2$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3Wh/\lambda Ta^2$	$10^2Mx/E\lambda Th^2$	$10^2My/E\lambda Th^2$	$10^2Mxy/E\lambda Th^2$
0.0	0.00	-10.61	0.00	-14.05
0.1	13.10	-17.78	-20.30	-21.34
0.2	21.04	-18.87	-21.77	- 4.84
0.3	20.12	-17.59	-17.75	6.13
0.4	18.91	-16.13	-14.04	6.40
0.5	13.83	-15.55	-12.66	0.00
0.6	18.91	-16.13	-14.04	- 6.40
0.7	20.12	-17.59	-17.75	- 6.13
0.8	21.04	-18.87	-21.77	4.84
0.9	13.10	-17.78	-20.30	21.34
1.0	0.00	-10.61	0.00	14.05

$a^2/hR_1=10$ $b/a=1.4$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\perp Ta^2$	$10^2 Mx/E\perp Th^2$	$10^2 My/E\perp Th^2$	$10^3 Mxy/E\perp Th^2$
0.0	0.00	-10.61	0.00	-10.72
0.1	12.66	-18.15	-21.66	-15.10
0.2	15.85	-18.25	-21.13	5.47
0.3	7.83	-16.06	-16.02	15.02
0.4	- 1.85	-14.14	-12.27	11.08
0.5	- 5.91	-13.43	-11.02	0.00
0.6	- 1.86	-14.14	-12.72	-11.08
0.7	7.83	-16.06	-16.02	-15.02
0.8	15.85	-18.25	-21.13	- 5.47
0.9	12.66	-18.15	-21.66	15.10
1.0	0.00	-10.61	0.00	10.72

$a^2/hR_1=10$ $b/a=1.6$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\perp Ta^2$	$10^2 Mx/E\perp Th^2$	$10^2 My/E\perp Th^2$	$10^3 Mxy/E\perp Th^2$
0.0	0.00	-10.61	0.00	- 8.75
0.1	12.37	-18.38	-22.60	- 9.94
0.2	10.46	-17.89	-20.07	13.17
0.3	- 3.94	-14.67	-14.61	20.09
0.4	-17.72	-12.60	-11.49	12.98
0.5	-23.08	-11.90	-10.60	0.00
0.6	-17.72	-12.60	-11.49	-12.98
0.7	- 3.94	-14.67	-14.61	-20.09
0.8	10.46	-17.49	-20.07	-13.07
0.9	12.37	-18.38	-22.60	9.94
1.0	0.00	-10.61	0.00	8.75

$a^2/hR_1=10$ $b/a=1.8$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\perp Ta^2$	$10^2 Mx/E\perp Th^2$	$10^2 My/E\perp Th^2$	$10^3 Mxy/E\perp Th^2$
0.0	0.00	-10.61	0.00	- 7.17
0.1	11.76	-18.44	-22.89	- 5.07
0.2	4.33	-16.69	-18.74	18.88
0.3	-15.41	-13.38	-13.46	22.16
0.4	-31.65	-11.39	-11.26	12.82
0.5	-37.36	-10.78	-10.82	0.00
0.6	-31.85	-11.39	-11.26	-12.82
0.7	-15.41	-13.38	-13.46	-22.16
0.8	4.33	-16.69	-18.74	-18.88
0.9	11.75	-18.45	-22.89	5.09
1.0	0.00	-10.61	0.00	7.24

$a^2/hR_1=10$ $b/a=2.0$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 w/\lambda T a^2$	$10^2 M_x/E\lambda Th^2$	$10^2 M_y/E\lambda Th^2$	$10^3 M_{xy}/E\lambda Th^2$
0.0	0.00	-10.61	0.00	- 5.73
0.1	10.66	-18.37	-22.98	- 6.32
0.2	-2 .62	-15.59	-17.33	22.29
0.3	-26.66	-12.19	-12.66	22.21
0.4	-43.86	-10.42	-11.31	11.48
0.5	-49.22	- 9.93	-11.26	0.00
0.6	-43.56	-10.42	-11.31	-11.48
0.7	-26.66	-12.19	-12.56	-22.21
0.8	- 2.63	-15.69	-17.33	-22.29
0.9	10.67	-18.37	-22.95	0.32
1.0	0.00	-10.61	0.00	5.85

$a^2/hR_1=15$ $b/a=1.0$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 w/\lambda T a^2$	$10^2 M_x/E\lambda Th^2$	$10^2 M_y/E\lambda Th^2$	$10^3 M_{xy}/E\lambda Th^2$
0.0	0.00	-10.51	0.00	2.28
0.1	3.11	-14.59	-16.76	- 8.61
0.2	5.94	-16.93	-19.14	- 1.59
0.3	4.01	-15.99	-16.03	4.61
0.4	2.31	-14.97	-13.89	4.43
0.5	1.12	-14.58	-12.32	0.00
0.6	2.31	-14.97	-13.89	- 4.43
0.7	4.91	-15.99	-16.03	- 4.61
0.8	5.94	-16.83	-19.14	1.59
0.9	3.11	-14.69	-16.76	8.61
1.0	0.00	-10.61	0.00	- 2.48

$a^2/hR_1=15$ $b/a=1.5$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 w/\lambda T a^2$	$10^2 M_x/E\lambda Th^2$	$10^2 M_y/E\lambda Th^2$	$10^3 M_{xy}/E\lambda Th^2$
0.0	0.00	-10.61	0.00	7.80
0.1	0.86	-16.13	-18.26	- 1.86
0.2	- 1.49	-16.11	-15.74	8.92
0.3	- 6.75	-14.39	-14.89	11.90
0.4	-16.08	-12.82	-11.79	9.60
0.5	-19.04	-12.26	-10.71	0.00
0.6	-16.08	-12.09	-11.79	- 9.60
0.7	- 6.80	-14.36	-14.89	-11.90
0.8	- 1.49	-16.11	-15.74	- 8.92
0.9	0.86	-16.13	-18.26	1.86
1.0	0.00	-10.61	0.00	- 7.80

$a^2/hR_1=15$ $b/a=1.4$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\alpha T a^2$	$10^2 Mx/E\alpha Th^2$	$10^2 My/E\alpha Th^2$	$10^3 Mxy/E\alpha Th^2$
0.0	0.00	-10.61		
0.1	- 1.66	-16.24	0.00	11.06
0.2	- 8.74	-15.26	-19.24	3.79
0.3	-21.83	-12.88	-17.91	16.36
0.4	-33.04	-11.15	-13.84	19.18
0.5	-37.28	-10.57	-10.86	11.86
0.6	-33.04	-11.15	-10.08	0.00
0.7	-21.83	-12.80	-10.86	-11.86
0.8	- 8.75	-15.26	-13.84	-19.18
0.9	- 1.66	-16.24	-17.91	-16.36
1.0	0.00	-10.61	-19.24	- 3.79
			0.00	-11.06

$a^2/hR_1=15$ $b/a=1.6$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\alpha T a^2$	$10^2 Mx/E\alpha Th^2$	$10^2 My/E\alpha Th^2$	$10^3 Mxy/E\alpha Th^2$
0.0	0.00	-10.61	0.00	13.38
0.1	- 3.98	-16.21	-19.75	7.72
0.2	-16.32	-14.30	-16.78	21.61
0.3	-34.34	-11.63	-12.47	21.48
0.4	-48.01	- 9.84	-10.60	12.09
0.5	-52.89	- 9.32	-10.21	0.00
0.6	-48.01	- 9.84	-10.60	-12.09
0.7	-34.33	-11.63	-12.47	-21.48
0.8	-16.32	-14.30	-16.78	-21.60
0.9	- 3.98	-16.21	-19.75	- 7.71
1.0	0.00	-10.61	0.00	-13.38

$a^2/hR_1=15$ $b/a=1.8$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\alpha T a^2$	$10^2 Mx/E\alpha Th^2$	$10^2 My/E\alpha Th^2$	$10^3 Mxy/E\alpha Th^2$
0.0	0.00	-10.61	0.00	15.26
0.1	- 6.54	-16.04	-19.90	11.78
0.2	-24.30	-13.27	-15.54	25.17
0.3	-46.22	-10.31	-11.67	21.71
0.4	-61.05	- 8.78	-10.65	11.10
0.5	-68.98	- 8.36	-10.60	0.00
0.6	-61.05	- 8.78	-10.65	-11.10
0.7	-46 .22	-10.31	-11.67	-21.70
0.8	-24.31	-13.27	-15.54	-25.16
0.9	- 6.55	-16.04	-19.90	-11.79
1.0	0.00	-10.61	0.00	-15.31

$a^2/hR_1=16$ $b/a=2.0$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\alpha T a^2$	$10^2 M_x/E\alpha T h^2$	$10^2 M_y/E\alpha T h^2$	$10^3 M_{xy}/E\alpha T h^2$
0.0	0.00	-10.61	0.00	16.79
0.1	- 9.36	-15.78	-19.80	15.53
0.2	-32.56	-12.24	-14.37	27.40
0.3	-57.37	- 9.23	-11.14	20.73
0.4	-72.41	- 7.89	-10.81	9.65
0.5	-77.10	- 7.86	-10.98	0.00
0.6	-72.41	- 7.89	-10.81	- 9.65
0.7	-57.37	- 9.23	-11.14	-20.72
0.8	-32.56	-12.24	-14.37	-27.39
0.9	- 9.38	-15.78	-19.81	-15.51
1.0	0.00	-10.61	0.00	-16.69

$a^2/hR_1=20$ $b/a=1.0$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\alpha T a^2$	$10^2 M_x/E\alpha T h^2$	$10^2 M_y/E\alpha T h^2$	$10^3 M_{xy}/E\alpha T h^2$
0.0	0.00	-10.61	0.00	15.47
0.1	- 4.69	-14.82	-15.33	5.98
0.2	- 8.49	-14.97	-16.88	9.87
0.3	-14.07	-13.73	-14.24	11.66
0.4	-19.09	-12.86	-11.73	7.67
0.5	-21.08	-12.12	-10.80	0.00
0.6	-19.09	-12.86	-11.73	- 7.67
0.7	-14.07	-13.73	-14.24	-11.66
0.8	- 8.49	-14.97	-16.88	- 9.87
0.9	- 4.69	-14.82	-15.33	- 5.98
1.0	0.00	-10.61	0.00	-15.47

$a^2/hR_1=20$ $b/a=1.2$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\alpha T a^2$	$10^2 M_x/E\alpha T h^2$	$10^2 M_y/E\alpha T h^2$	$10^3 M_{xy}/E\alpha T h^2$
0.0	0.00	-10.61	0.00	23.45
0.1	- 8.34	-14.89	-16.55	11.72
0.2	-17.13	-14.11	-16.36	17.73
0.3	-28.19	-12.16	-12.90	17.96
0.4	-37.05	-10.71	-10.50	10.83
0.5	-40.36	-10.21	- 9.75	0.00
0.6	-37.05	-10.71	-10.50	-10.83
0.7	-28.19	-12.16	-12.90	-17.96
0.8	-17.14	-14.11	-16.36	-17.73
0.9	- 8.33	-14.89	-16.55	-12.72
1.0	0.00	-10.61	0.00	-23.45

$a^2/hR_1=20$ $b/a=1.4$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\alpha Ta^2$	$10^2 Mx/E\alpha Th^2$	$10^2 My/E\alpha Th^2$	$10^3 Mxy/E\alpha Th^2$
0.0	0.00	-10.61	0.00	26.77
0.1	-11.91	-14.81	-17.25	16.13
0.2	-26.83	-13.12	-15.45	23.07
0.3	-41.69	-10.74	-11.86	20.91
0.4	-53.26	- 9.27	-10.13	11.65
0.5	-57.36	- 8.81	- 9.72	0.00
0.6	-53.26	- 9.27	-10.13	-11.65
0.7	-41.69	-10.74	-11.86	-20.91
0.8	-26.83	-13.12	-15.45	-23.07
0.9	-11.91	-14.80	-17.25	-16.13
1.0	0.00	-10.61	0.00	-26.78

$a^2/hR_1=20$ $b/a=1.6$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\alpha Ta^2$	$10^2 Mx/E\alpha Th^2$	$10^2 My/E\alpha Th^2$	$10^3 Mxy/E\alpha Th^2$
0.0	0.00	-10.61	0.00	29.23
0.1	-15.55	-14.60	-17.55	19.86
0.2	-34.66	-12.07	-14.37	26.82
0.3	-54.38	- 9.47	-11.11	21.52
0.4	-67.44	- 8.11	-10.17	11.04
0.5	-71.81	- 7.73	-10.10	0.00
0.6	-67.44	- 8.11	-10.17	-11.04
0.7	-54.38	- 9.47	-11.11	-21.52
0.8	-34.66	-12.07	-14.37	-26.81
0.9	-15.55	-14.59	-17.54	-19.85
1.0	0.00	-10.61	0.00	-29.26

$a^2/hR_1=20$ $b/a=1.8$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\alpha Ta^2$	$10^2 Mx/E\alpha Th^2$	$10^2 My/E\alpha Th^2$	$10^3 Mxy/E\alpha Th^2$
0.0	0.00	-10.61	0.00	31.11
0.1	-19.29	-14.29	-17.56	23.08
0.2	-43.49	-11.03	-13.33	28.50
0.3	-66.10	- 8.36	-10.64	20.76
0.4	-79.74	- 7.15	-10.38	9.82
0.5	-84.05	- 6.84	-10.49	0.00
0.6	-79.73	- 7.15	-10.38	- 9.83
0.7	-66.09	- 8.36	-10.64	-20.77
0.8	-43.46	-11.04	-13.34	-28.49
0.9	-19.30	-14.30	-17.58	-23.08
1.0	0.00	-10.60	0.00	-30.94

$a^2/hR_1=20$ $b/a=2.0$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\lambda Ta^2$	$10^2 Mx/E\lambda Th^2$	$10^2 My/E\lambda Th^2$	$10^3 Mxy/E\lambda Th^2$
0.0	0.00	-10.61	0.00	32.56
0.1	-23.11	-13.92	-17.36	25.87
0.2	-52.15	-10.02	-12.42	29.34
0.3	-76.77	- 7.14	-10.41	19.30
0.4	-90.39	- 6.34	-10.55	8.83
0.5	-94.51	- 6.06	-10.77	0.00
0.6	-90.39	- 6.34	-10.55	- 8.83
0.7	-76.75	- 7.41	-10.40	-19.32
0.8	-52.09	-10.04	-12.41	-29.40
0.9	-22.96	-13.94	-17.40	-26.92
1.0	0.00	-10.68	0.00	-31.89

<u>Two Edges somply supported and two edges clamped</u>

$a^2/hR_1=5$ $b/a=1.0$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\lambda Ta^2$	$10^2 Mx/E\lambda Th^2$	$10^2 My/E\lambda Th^2$	$10^3 Mxy/E\lambda Th^2$
0.0	0.00	- 4.27	21.12	0.00
0.1	13.02	-14.57	- 9.61	-41.43
0.2	34.24	-19.71	-20.96	-40.14
0.3	50.92	-21.80	-23.33	-26.37
0.4	60.42	-22.40	-22.74	-12.23
0.5	63.40	-22.49	-22.23	0.00
0.6	60.42	-22.40	-22.74	12.23
0.7	50.92	-21.80	-23.33	26.37
0.8	34.24	-19.71	-20.96	40.14
0.9	13.02	-14.57	- 9.61	41.43
1.0	0.00	- 4.27	21.12	0.00

$a^2/hR_1=5$ $b/a=1.2$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\lambda Ta^2$	$10^2 Mx/E\lambda Th^2$	$10^2 My/E\lambda Th^2$	$10^3 Mxy/E\lambda Th^2$
0.0	0.00	-5.50	17.03	0.00
0.1	14.74	-16.21	-14.64	-35.91
0.2	33.98	-20.12	-22.37	-25.62
0.3	44.44	-20.55	-21.03	-10.26
0.4	47.72	-19.96	-18.09	- 2.08
0.5	48.17	-19.62	-16.81	0.00
0.6	47.72	-19.96	-18.09	2.08
0.7	44.44	-20.55	-21.03	10. 26
0.8	43.98	-20. 12	-22.37	25.62
0.9	14.74	-16.21	-14.64	35.91
1.0	0.00	- 5.50	17.03	0.00

$a^2/hR_1=5$ $b/a=1.4$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3W/\alpha Ta^2$	$10^2M_x/E\alpha Th^2$	$10^2M_y/E\alpha Th^2$	$10^3M_{xy}/E\alpha Th^2$
0.0	0.00	- 6.04	15.21	0.00
0.1	16.70	-17.41	-18.08	-31.54
0.2	32.98	-20.21	-22.95	-13.19
0.3	35.87	-19.32	-19.21	2.87
0.4	32.13	-17.79	-15.14	5.80
0.5	29.85	-17.14	-13.58	0.00
0.6	32.14	-17.80	-15.14	- 5.80
0.7	35.87	-19.32	-19.21	- 2.88
0.8	32.99	-20.21	-22.96	13.19
0.9	16.70	-17.41	-18.08	33.04
1.0	0.00	- 6.04	15.21	0.00

$a^2/hR_1=5$ $b/a=1.6$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3W/\alpha Ta^2$	$10^2M_x/E\alpha Th^2$	$10^2M_y/E\alpha Th^2$	$10^3M_{xy}/E\alpha Th^2$
0.0	0.00	- 6.28	14.51	0.00
0.1	18.90	-18.28	-20.40	-27.97
0.2	31.70	-20.05	-22.80	- 3.28
0.3	27.01	-18.13	-17.67	12.08
0.4	17.04	-16.05	-13.46	10.70
0.5	12.48	-15.24	-12.02	0.00
0.6	17.30	-16.05	-13.46	-10.70
0.7	27.00	-18.13	-17.67	-12.08
0.8	31.69	-20.05	-22.79	3.27
0.9	18.89	-18.28	-20.39	27.96
1.0	0.00	- 6.25	14.54	0.00

$a^2/hR_1=5$ $b/a=1.8$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3W/\alpha Ta^2$	$10^2M_x/E\alpha Th^2$	$10^2M_y/E\alpha Th^2$	$10^3M_{xy}/E\alpha Th^2$
0.0	0.00	- 6.37	14.11	0.00
0.1	20.99	-18.93	-21.97	-24.42
0.2	29.66	-19.66	-22.03	4.61
0.3	18.0	-16.98	-16.28	17.63
0.4	3.83	-14.69	-12.64	12.80
0.5	- 2.44	-13.88	-11.57	0.00
0.6	3.82	-14.69	-12.64	-12.78
0.7	18.05	-16.98	-16.29	-17.59
0.8	29.59	-19.65	-22.03	- 4.56
0.9	20.92	-18.91	-21.92	24.41
1.0	0.00	- 6.34	14.28	0.00

$a^2/hR_1=5$ $b/a=2.0$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3Wh/\lambda Ta^2$	$10^2Mx/E\lambda Th^2$	$10^2My/E\lambda Th^2$	$10^3Mxy/E\lambda Th^2$
0.0	0.00	- 6.49	13.72	0.00
0.1	22.68	-19.38	-23.00	-20.48
0.2	26.43	-19.05	-20.88	10.93
0.3	8.98	-15.87	-15.06	20.24
0.4	- 6.20	-13.64	-12.33	12.74
0.5	-14.68	-12.92	-11.74	0.00
0.6	- 8.23	-13.64	-12.35	-12.70
0.7	8.87	-15.86	-15.08	-20.11
0.8	26.17	-19.03	-20.88	-10.71
0.9	22.30	-19.31	-22.87	20.57
1.0	0.00	- 6.38	14.26	0.00

$a^2/hR_1=10$ $b/a=1.0$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3Wh/\lambda Ta^2$	$10^2Mx/E\lambda Th^2$	$10^2My/E\lambda Th^2$	$10^3Mxy/E\lambda Th^2$
0.0	0.00	- 7.56	10.16	0.00
0.1	8.40	-15.40	-13.64	-25.31
0.2	20.24	-18.39	-20.37	-20.08
0.3	27.61	-18.88	-19.99	-09.82
0.4	30.64	-18.54	-18.03	- 3.14
0.5	31.33	-18.32	-17.12	0.00
0.6	30.64	-18.54	-18.08	3.14
0.7	27.61	-18.88	-19.99	9.82
0.8	20.24	-18.39	-20.37	20.08
0.9	8.40	-15.40	-13.64	25.31
1.0	0.00	- 7.56	10.16	0.00

$a^2/hR_1=10$ $b/a=1.2$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3Wh/\lambda Ta^2$	$10^2Mx/E\lambda Th^2$	$10^2My/E\lambda Th^2$	$10^3Mxy/E\lambda Th^2$
0.0	0.00	- 8.58	6.75	0.00
0.1	8.80	-16.59	-17.51	-19.19
0.2	16.93	-18.31	-21.02	- 6.71
0.3	17.35	-17.40	-17.87	3.87
0.4	14.23	-16.12	-14.46	5.09
0.5	12.55	-15.58	-13.14	0.00
0.6	14.23	-16.12	-14.46	- 5.09
0.7	17.35	-17.40	-17.87	- 3.87
0.8	16.93	-18.31	-21.02	6.71
0.9	8.80	-16.59	-17.51	19.19
1.0	0.00	- 8.58	6.75	0.00

$a^2/hR_1=10$ $b/a=1.4$ $x/a=0.5$ $R_1/R_2=0.6$

Y/b	$10^3 wh/\alpha T a^2$	$10^2 M_x/E\alpha Th^2$	$10^2 M_y/E\alpha Th^2$	$10^3 M_{xy}/E\alpha Th^2$
0.0	0.00	- 9.07	5.11	0.00
0.1	9.23	-17.33	-19.88	-14.22
0.2	12.96	-17.92	-20.82	3.78
0.3	6.15	-15.97	-16.19	13.47
0.4	- 2.72	-14.14	-12.81	10.31
0.5	- 6.81	-13.45	-11.25	0.00
0.6	- 2.72	-14.14	-12.81	-10.31
0.7	6.15	-15.97	-16.19	-13.47
0.8	12.96	-17.92	-20.82	- 3.78
0.9	9.23	-17.33	-19.88	14.22
1.0	0.00	- 9.07	5.11	0.00

$a^2/hR_1=10$ $b/a=1.6$ $x/a=0.5$ $R_1/R_2=0.6$

Y/b	$10^3 wh/\alpha T a^2$	$10^2 M_x/E\alpha Th^2$	$10^2 M_y/E\alpha Th^2$	$10^3 M_{xy}/E\alpha Th^2$
0.0	0.00	- 9.36	4.16	0.00
0.1	9.52	-17.78	-21.30	- 9.74
0.2	8.35	-17.29	-19.99	11.74
0.3	- 5.04	-14.63	-14.77	19.09
0.4	-18.27	-12.59	-11.61	12.58
0.5	-23.45	-11.89	-10.69	0.00
0.6	-18.27	-12.59	-11.61	-12.58
0.7	- 5.03	-14.63	-14.77	-19.09
0.8	8.35	-17.29	-19.99	-11.74
0.9	9.53	-17.78	-21.30	9.73
1.0	0.00	- 9.36	4.16	0.00

$a^2/hR_1=10$ $b/a=1.8$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 wh/\alpha T a^2$	$10^2 M_x/E\alpha Th^2$	$10^2 M_y/E\alpha Th^2$	$10^3 M_{xy}/E\alpha Th^2$
0.0	0.00	- 9.58	3.41	0.00
0.1	9.42	-18.01	-22.09	- 5.27
0.2	2.82	-16.47	-18.77	17.72
0.3	-16.13	-13.36	-13.59	21.57
0.4	-31.96	-11.38	-11.32	12.66
0.5	-37.70	-10.76	-10.84	0.00
0.6	-31.96	-11.38	-11.32	-12.65
0.7	-16.13	-13.36	-13.59	-21.57
0.8	2.81	-16.47	-18.77	-17.71
0.9	9.40	-18.01	-22.08	5.28
1.0	0.00	- 9.58	3.43	0.00

$a^2/hR_1=10$ $b/a=2.0$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3W/\alpha Ta^2$	$10^2Mx/E\alpha Th^2$	$10^2My/E\alpha Th^2$	$10^3Mxy/E\alpha Th^2$
0.0	0.00	- 9.79	2.73	0.00
0.1	8.82	-18.07	-22.44	- 0.73
0.2	- 3.65	-15.53	-17.40	22.12
0.3	-27.02	-12.18	-12.65	21.90
0.4	-43.88	-10.40	-11.33	11.46
0.5	-49.53	- 9.91	-11.25	0.00
0.6	-43.88	-10.40	-11 .33	-11.44
0.7	-27.03	-12.18	-12.65	-21.90
0.8	- 3.68	-15.53	-17.40	-22.12
0.9	8.83	-18.06	-22.41	0.71
1.0	0.00	- 9.78	2.79	0.00

$a^2/hR_1=15$ $b/a=1.0$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3W/\alpha Ta^2$	$10^2Mx/E\alpha Th^2$	$10^2My/E\alpha Th^2$	$10^3Mxy/E\alpha Th^2$
0.0	0.00	-10.98	- 1.22	0.00
0.1	3.72	-16.11	-17.31	- 9.23
0.2	6.46	-16.93	-19.30	- 1.12
0.3	5.24	-16.03	-16.61	4.98
0.4	2.60	-14.97	-13.81	4.68
0.5	1.34	-14.54	-12.72	0.00
0.6	2.60	-14.97	-13.81	- 4.65
0.7	5.24	-16.03	-16.61	- 4.98
0.8	6.46	-16.93	-19.30	1.12
0.9	3.72	-16.11	-17.31	9.23
1.0	0.00	-10.98	- 1.23	0.00

$a^2/hR_1=15$ $b/a=1.2$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3W/\alpha Ta^2$	$10^2Mx/E\alpha Th^2$	$10^2My/E\alpha Th^2$	$10^3Mxy/E\alpha Th^2$
0.0	0.00	-11.85	- 4.13	0.00
0.1	2.76	-16.77	-19.79	- 2.63
0.2	0.41	-16.35	-19.03	10.16
0.3	- 7.63	-14.41	-14.74	15.00
0.4	-15.51	-12.80	-11.86	10.13
0.5	-16.66	-12.22	-10.48	0.00
0.6	-15.51	-12.80	-11.86	-10.13
0.7	- 7.63	-14.41	-14.74	-15.00
0.8	0.41	-16.35	-19.03	-10.16
0.9	2.76	-16.77	-19.79	2.63
1.0	0.00	-11.85	- 4.13	0.00

$a^2/hR_1=15$ $b/a=1.4$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3wh/\alpha Ta^2$	$10^2M_x/E\alpha Th^2$	$10^2M_y/E\alpha Th^2$	$10^3M_{xy}/E\alpha Th^2$
0.0	0.00	-12.36	- 5.84	0.00
0.1	1.54	-17.03	-20.99	2.98
0.2	- 6.36	-15.50	-18.04	18.23
0.3	-20.62	-12.90	-13.30	20.53
0.4	-32.48	-11.14	-10.66	12.42
0.5	-36.90	-10.55	- 9.92	0.00
0.6	-32.48	-11.14	-10.66	-12.42
0.7	-20.62	-12.90	-13.29	-20.52
0.8	- 6.36	-15.03	-18.04	-18.23
0.9	1.54	-17.03	-20.99	- 2.98
1.0	0.00	-12.36	- 5.84	0.00

$a^2/hR_1=15$ $b/a=1.6$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3wh/\alpha Ta^2$	$10^2M_x/E\alpha Th^2$	$10^2M_y/E\alpha Th^2$	$10^3M_{xy}/E\alpha Th^2$
0.0	0.00	-12.73	- 7.07	0.00
0.1	- 0.10	-17.03	-21.43	8.13
0.2	-13.86	-14.48	-16.69	23.79
0.3	-33.23	-11.53	-12.18	22.62
0.4	-47.45	- 9.85	-10.47	12.41
0.5	-52.42	- 9.36	-10.16	0.00
0.6	-47.45	- 9.85	-10.47	-12.41
0.7	-33.23	-11.53	-12.18	-22.62
0.8	-13.86	-14.48	-16.69	-23.79
0.9	- 0.09	-17.03	-21.43	- 8.13
1.0	0.00	-12.73	- 7.07	0.00

$a^2/hR_1=15$ $b/a=1.8$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3wh/\alpha Ta^2$	$10^2M_x/E\alpha Th^2$	$10^2M_y/E\alpha Th^2$	$10^3M_{xy}/E\alpha Th^2$
0.0	0.00	-13.06	- 8.06	0.00
0.1	- 2.21	-16.86	-21.40	12.95
0.2	-21.98	-13.39	-15.29	27.43
0.3	-45.27	-10.31	-11.40	22.53
0.4	-60.44	- 8.81	-10.59	11.90
0.5	-65.40	- 8.42	-10.62	0.00
0.6	-60.44	- 8.82	-10.59	-11.90
0.7	-45.26	-10.31	-11.39	-22.53
0.8	-21.99	-13.39	-15.29	-27.43
0.9	- 2.20	-16.85	-21.40	-12.95
1.0	0.00	-13.05	- 8.09	0.00

$a^2/hR_1=15$ $b/a=2.0$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\alpha T a^2$	$10^2 Mx/E\alpha Th^2$	$10^2 My/E\alpha Th^2$	$10^3 Mxy/E\alpha Th^2$
0.0	0.00	-13.28	- 8.89	0.00
0.1	- 4.76	-16.53	-21.04	17.41
0.2	-30.49	-12.30	-14.00	29.55
0.3	-56.56	- 9.23	-10.91	21.23
0.4	-71.74	- 7.95	-10.80	9.60
0.5	-76.38	- 7.63	-11.03	0.00
0.6	-71.75	- 7.95	-10.80	- 9.60
0.7	-56.57	- 9.23	-10.91	-21.23
0.8	-30.51	-12.30	-14.00	-29.53
0.9	- 4.80	-16.53	-21.05	-17.39
1.0	0.00	-13.27	- 8.88	0.00

$a^2/hR_1=20$ $b/a=1.0$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\alpha T a^2$	$10^2 Mx/E\alpha Th^2$	$10^2 My/E\alpha Th^2$	$10^3 Mxy/E\alpha Th^2$
0.0	0.00	-13.86	-10.83	0.00
0.1	- 0.08	-16.53	-19.76	3.51
0.2	- 4.27	-15.63	-17.91	12.66
0.3	-11.43	-13.84	-13.91	14.75
0.4	-17.67	-12.46	-11.02	9.39
0.5	-20.09	-11.97	-10.03	0.00
0.6	-17.67	-12.46	-11.02	- 9.39
0.7	-11.43	-13.84	-13.91	-14.75
0.8	- 4.27	-15.63	-17.91	-12.66
0.9	- 0.08	-16.53	-19.76	- 3.51
1.0	0.00	-13.86	-10.83	0.00

$a^2/hR_1=20$ $b/a=1.2$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\alpha T a^2$	$10^2 Mx/E\alpha Th^2$	$10^2 My/E\alpha Th^2$	$10^3 Mxy/E\alpha Th^2$
0.0	0.00	-14.71	-13.66	0.00
0.1	- 2.24	-16.73	-21.02	10.60
0.2	-12.43	-14.63	-16.81	21.78
0.3	-25.77	-12.18	-12.30	21.17
0.4	-36.01	-10.62	- 9.91	12.27
0.5	-39.75	-10.12	- 9.24	0.00
0.6	-36.01	-10.62	- 9.91	-12.27
0.7	-25.77	-12.18	-12.30	-21.17
0.8	-12.43	-14.63	-16.81	-21.78
0.9	- 2.24	-16.73	-21.02	-10.60
1.0	0.00	-14.71	-13.66	0.00

$a^2/hR_1=20$ $b/a=1.4$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\alpha T a^2$	$10^2 M_x/E\alpha Th^2$	$10^2 M_y/E\alpha Th^2$	$10^3 M_{xy}/E\alpha Th^2$
0.0	0.00	-15.28	-15.58	0.00
0.1	- 4.90	-16.57	-21.21	16.71
0.2	-21.32	-13.45	-15.31	27.62
0.3	-39.72	-10.70	-11.18	23.46
0.4	-52.39	- 9.23	-9 .77	12.45
0.5	-56.73	- 8.81	- 9.55	0.00
0.6	-52.39	- 9.23	- 9.78	-12.45
0.7	-39.72	-10.70	-11.18	-23.46
0.8	-21.32	-13.45	-15.31	-27.63
0.9	- 4.90	-16.57	-21.21	-16.71
1.0	0.00	-15.29	-15.58	0.00

$a^2/hR_1=20$ $b/a=1.6$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\alpha T a^2$	$10^2 M_x/E\alpha Th^2$	$10^2 M_y/E\alpha Th^2$	$10^3 M_{xy}/E\alpha Th^2$
0.0	0.00	-15.72	-17.02	0.00
0.1	- 8.05	-16.19	-20.82	22.08
0.2	-30.68	-12.23	-13.80	31.02
0.3	-52.85	- 9.41	-10.48	23.22
0.4	-66.57	- 8.14	-10.01	11.27
0.5	-70.99	- 7.80	-10.00	0.00
0.6	-66.57	- 8.14	-10.01	-11.27
0.7	-52.85	- 9.40	-10.48	-23.22
0.8	-30.68	-12.23	-13.81	-31.02
0.9	- 8.06	-16.20	-20.83	-22.06
1.0	0.00	-15.72	-17.03	0.00

$a^2/hR_1=20$ $b/a=1.8$ $x/a=0.5$ $R_1/R_2=0.5$

Y/b	$10^3 Wh/\alpha T a^2$	$10^2 M_x/E\alpha Th^2$	$10^2 M_y/E\alpha Th^2$	$10^3 M_{xy}/E\alpha Th^2$
0.0	0.00	-16.05	-18.14	0.00
0.1	-11.63	-15.66	-20.10	26.71
0.2	-40.18	-11.06	-12.52	32.58
0.3	-64.92	- 8.32	-10.15	21.71
0.4	-78.80	- 7.22	-10.52	9.73
0.5	-83.04	- 6.95	-10.58	0.00
0.6	-78.80	- 7.22	-10.52	- 9.73
0.7	-64.92	- 8.32	-10.15	-21.70
0.8	-40.19	-11.06	-12.51	-32.57
0.9	-11.64	-15.67	-20.07	-26.75
1.0	0.00	-16.03	-18.04	0.00

$a^2/hR_1=20$	$b/a=2.0$	$x/a=0.5$	$R_1/R_2=0.5$	
Y/b	$10^3 Wh/\alpha Ta^2$	$10^2 Mx/E\alpha Th^2$	$10^2 My/E\alpha Th^2$	$10^3 Mxy/E\alpha Th^2$
0.0	0.00	-16.31	-19.00	0.00
0.1	-15.54	-15.09	-19.19	30.58
0.2	-49.49	- 9.99	-11.52	32.78
0.3	-75.79	- 7.40	-10.08	19.71
0.4	-89.38	- 6.43	-10.58	8.33
0.5	-93.37	- 6.20	-10.86	0.00
0.6	-89.37	- 6.43	-10.58	- 8.34
0.7	-75.76	- 7.40	-10.08	-19.72
0.8	-49.47	-10.00	-11.54	-32.76
0.9	-15.57	-15.08	-19.16	-30.47
1.0	0.00	-16.22	-18.66	0.00

Tabulation of Deflection and Moments for various values of a^2/hR_1 and b/a ratio constant.

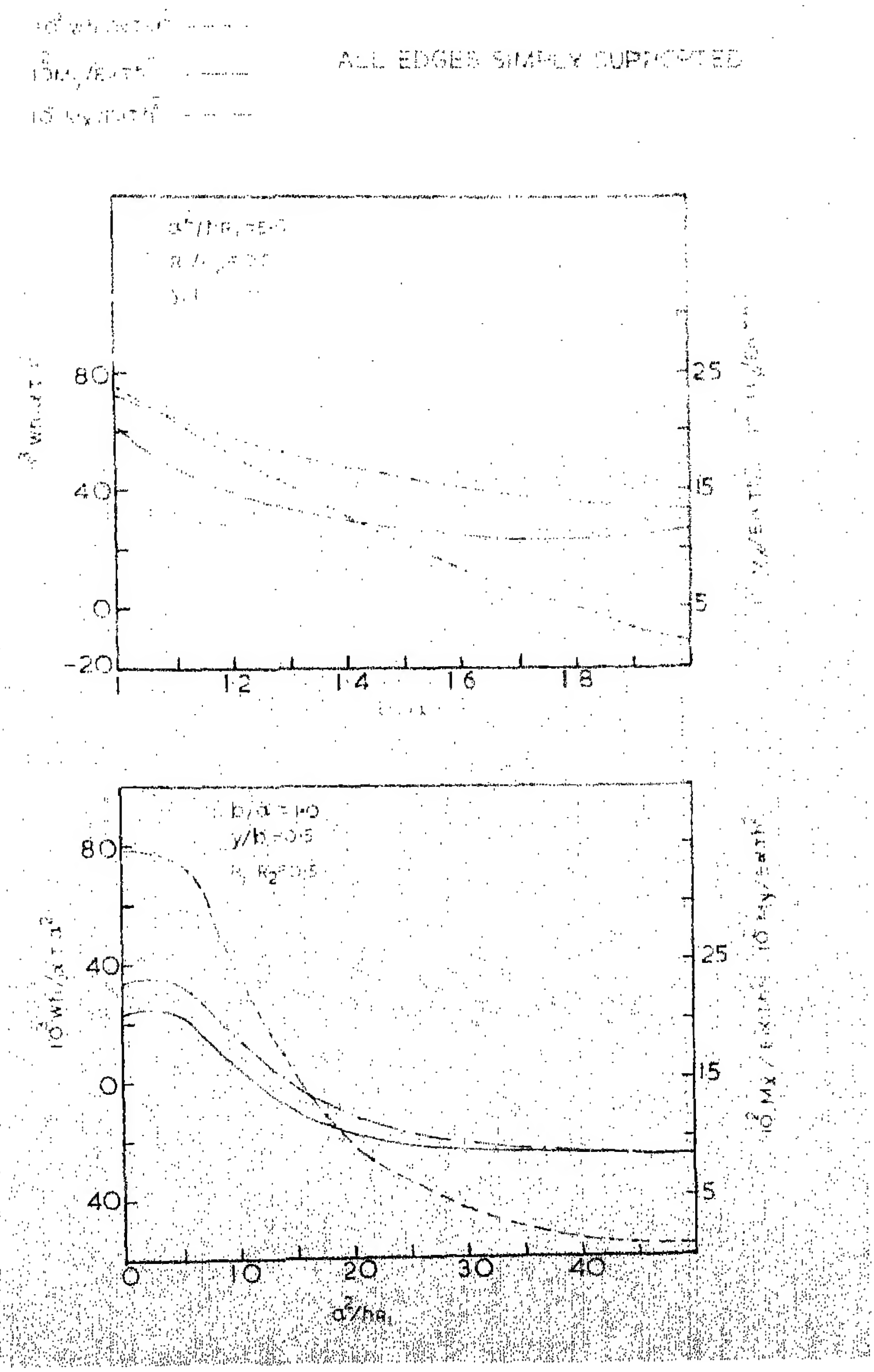
<u>All edges simply supported.</u>

$b/a=1.0$	$y/b=0.5$	$x/a=0.5$	$R_1/R_2=0.5$
a^2/hR_1	$10^3 Wh/\alpha Ta^2$	$10^2 Mx/E\alpha Th^2$	$10^2 My/E\alpha Th^2$
5	53.75	-19.68	-15.48
10	34.80	-18.36	-16.24
15	1.12	-14.88	-12.82
20	-21.08	-12.12	-10.80
25	-35.06	-10.64	- 9.72
30	-44.01	- 9.74	- 9.17
35	-49.91	- 9.17	- 8.91
40	-53.95	- 8.81	- 8.79
45	-56.80	- 8.56	- 8.74
50	-58.88	- 8.39	- 8.74

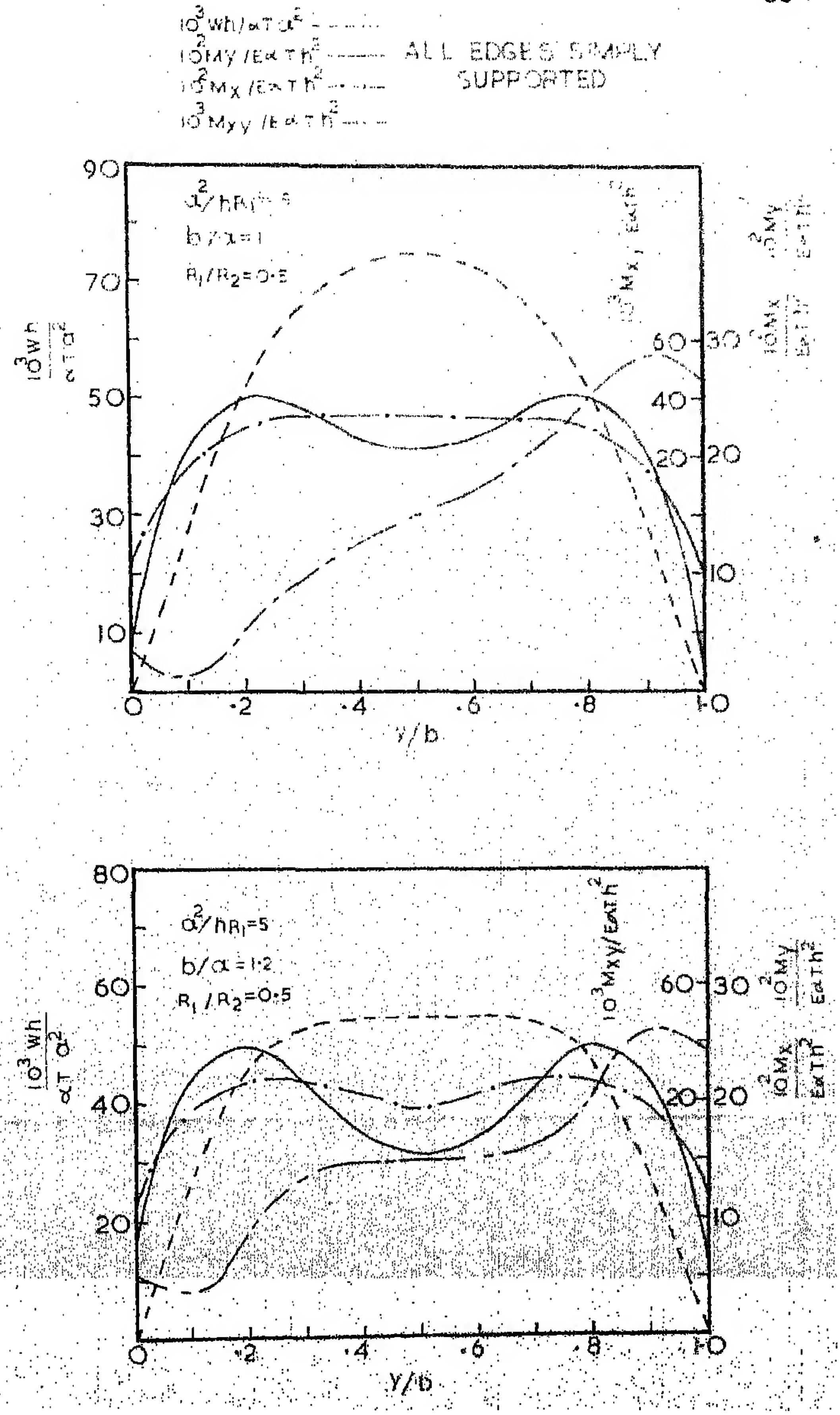
<u>Two Edges simply supported and two edges clamped</u>

$b/a=1.0$	$y/b=0.5$	$x/a=0.5$	$R_1/R_2=0.5$
a^2/hR_1	$10^3 Wh/\alpha Ta^2$	$10^2 Mx/E\alpha Th^2$	$10^2 My/E\alpha Th^2$
5	63.40	-22.49	-22.23
10	31.33	-18.32	-17.12
15	1.34	-14.54	-12.72
20	-20.09	-11.97	-10.03
25	-34.26	-10.40	- 8.68
30	-43.51	- 9.48	- 8.16
35	-49.64	- 8.95	- 8.07
40	-53.79	- 8.64	- 8.19
45	-56.68	- 8.46	- 8.38
50	-58.76	- 8.35	- 8.56

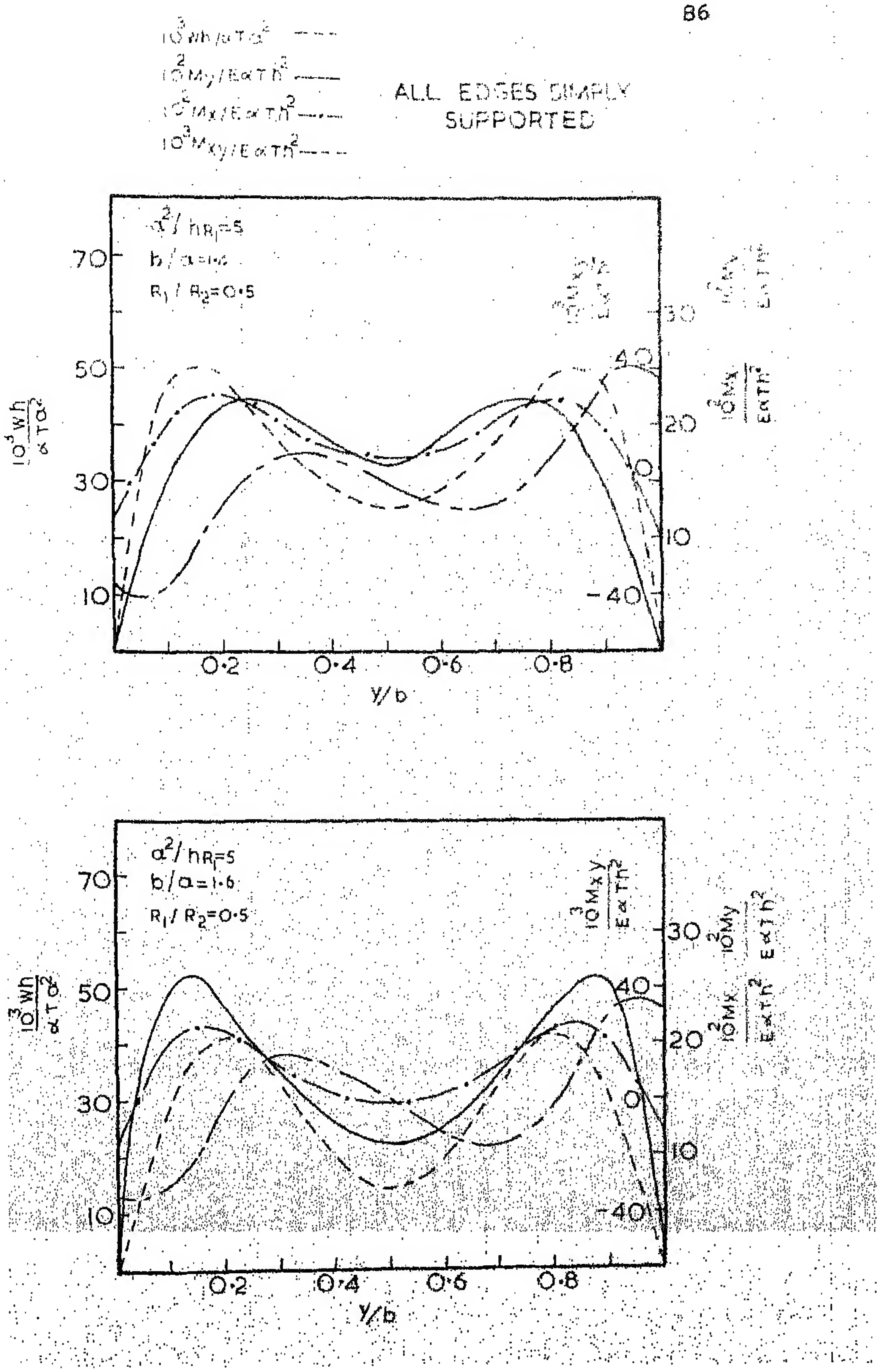
APPENDIX 3E

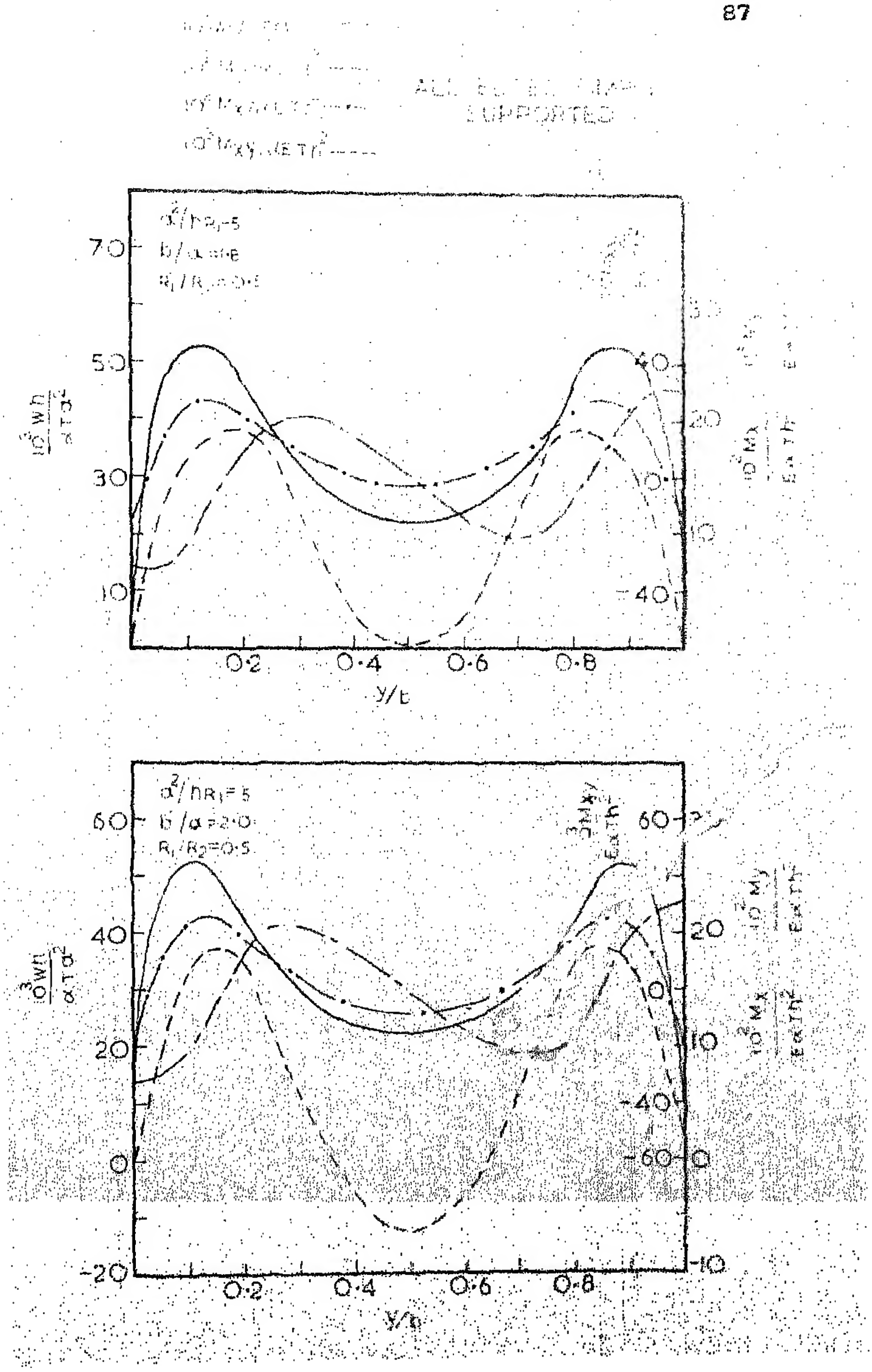


ALL EDGES SIMPLY SUPPORTED
80
40
0
-20
25
15
5
1
1·2
1·4
1·6
1·8
b/a = 1·0
y/b = 0·5
80
40
0
40
25
15
5
0
10
20
30
40

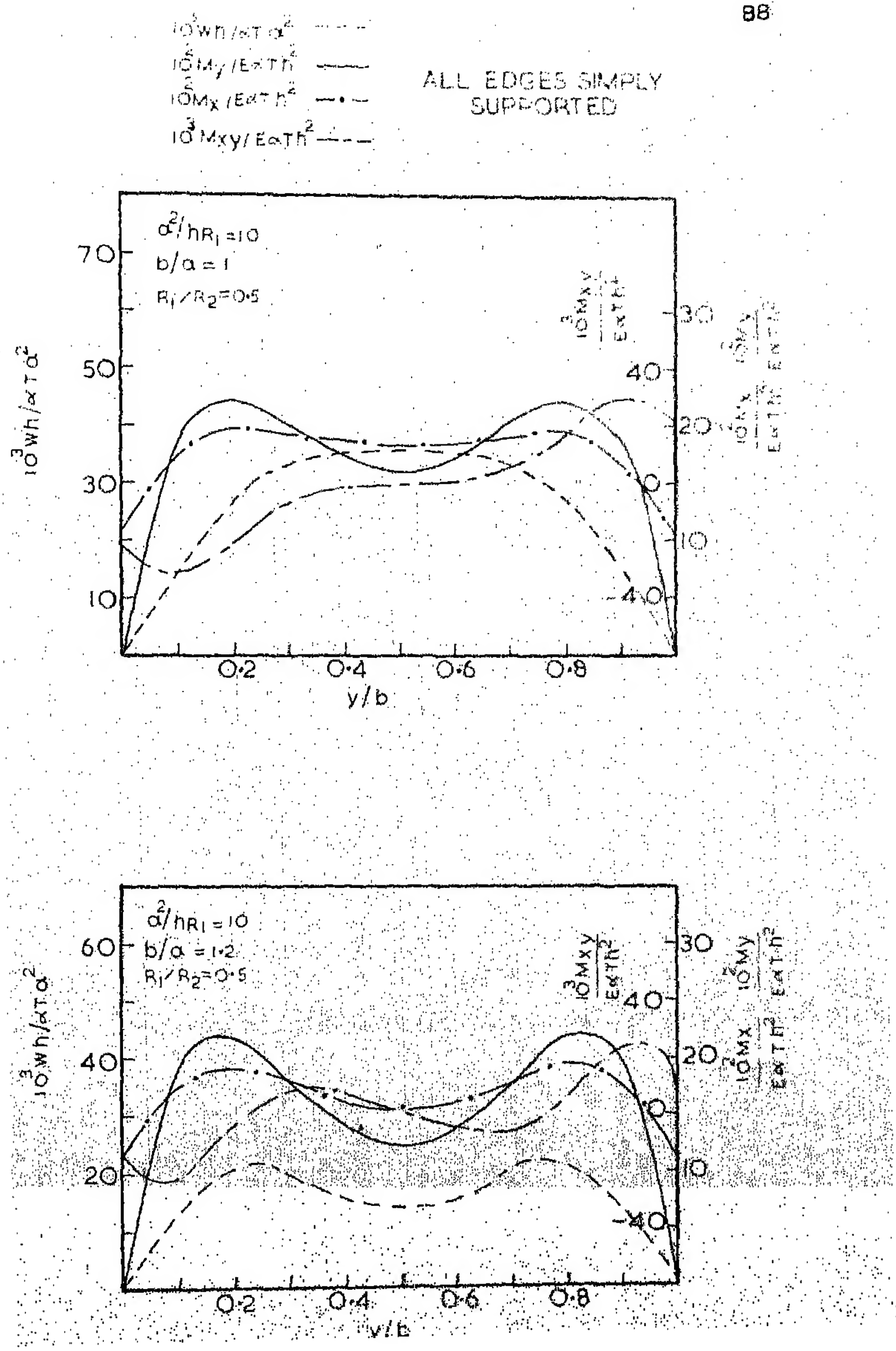

ALL EDGES SIMPLY SUPPORTED
$a^2/hR_1=5$
$b/a=1$
$R_1/R_2=0.5$
$a^2/hR_1=5$
$b/a=1.2$
$R_1/R_2=0.5$
y/b

ALL EDGES SIMPLY SUPPORTED



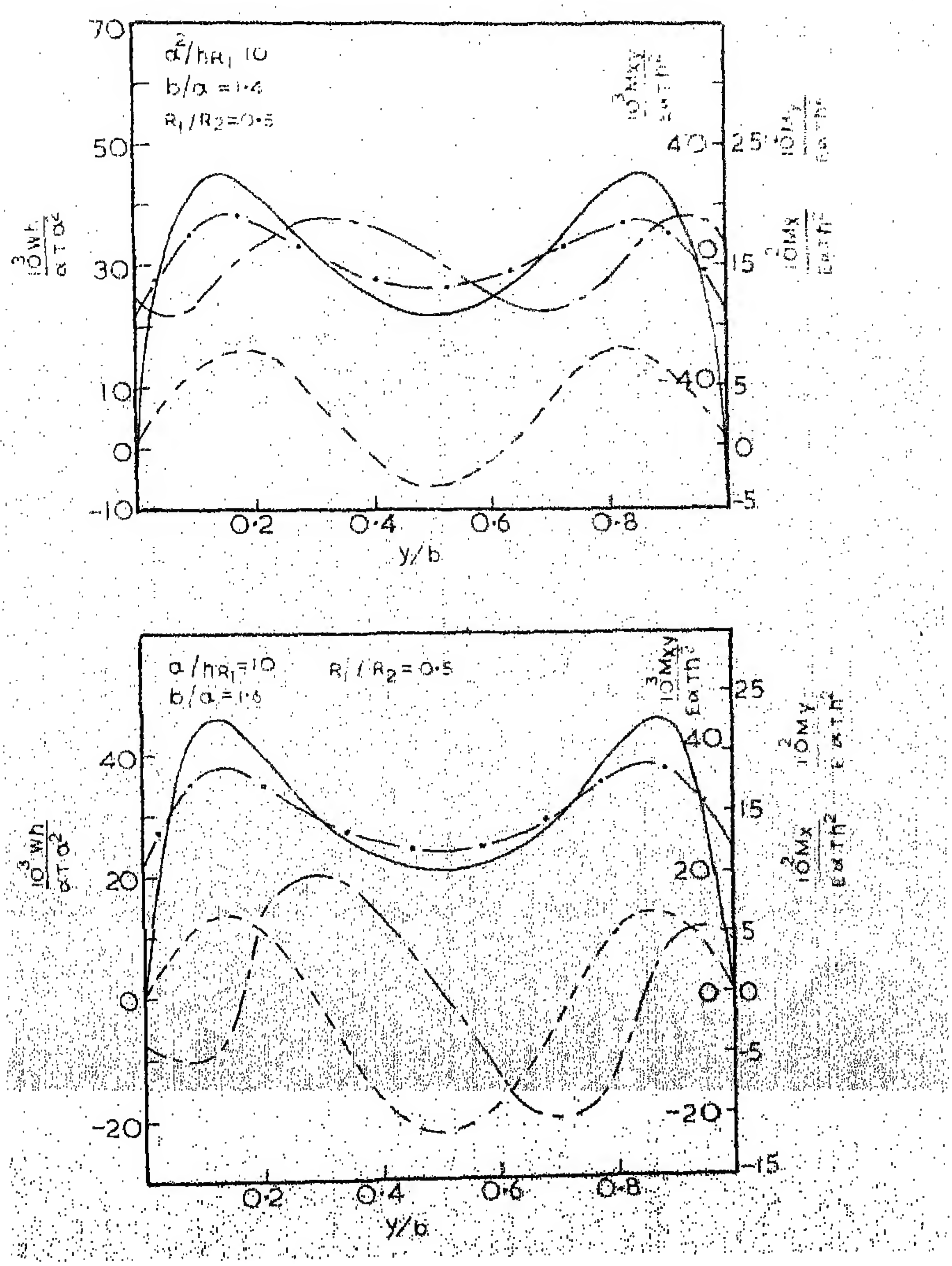


$a^2/hR_1=5$
$b/a=1\cdot0$
$R_1/R_2=0\cdot5$
$\frac{10^3wh}{\alpha T a^2}$
$\frac{10^2M_x}{E\alpha Th^2}$
y/b
$a^2/hR_1=5$
$b/a=2\cdot0$
$R_1/R_2=0\cdot5$
$\frac{10^3M_{xy}}{E\alpha Th^2}$
$\frac{10^2M_y}{E\alpha Th^2}$
y/b

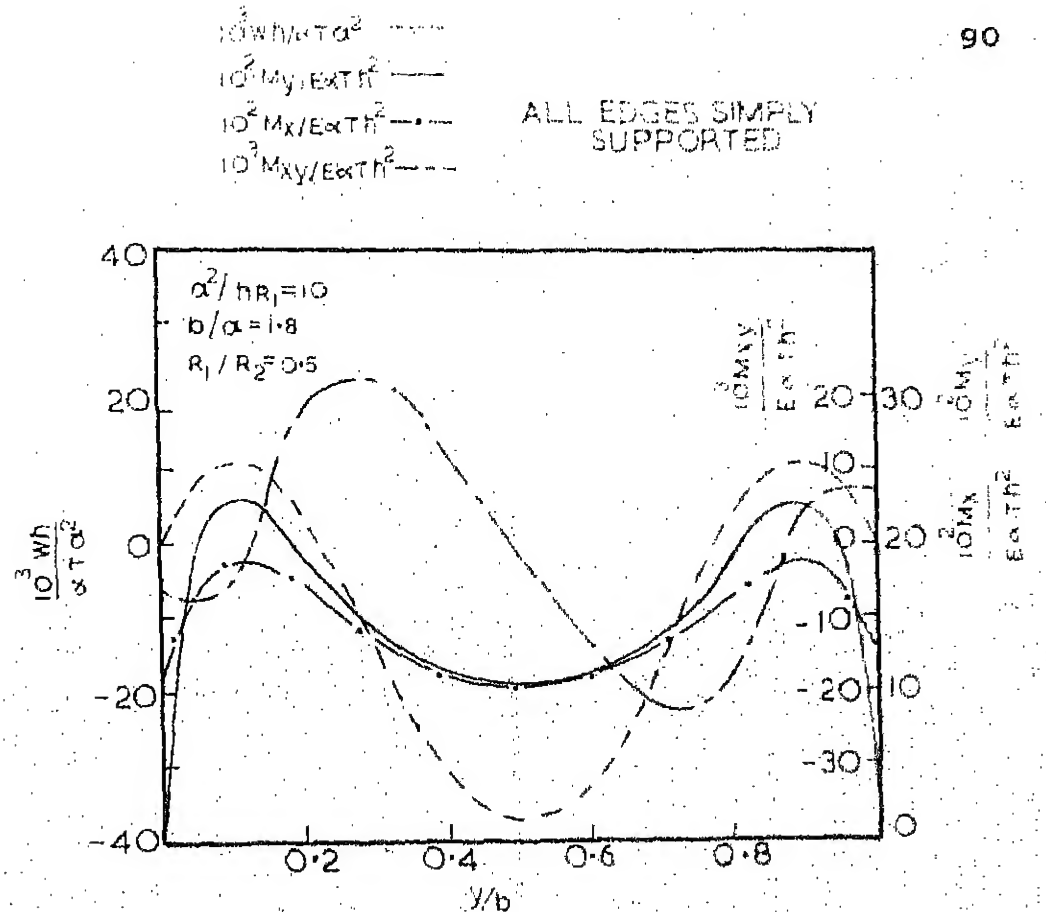

$10^3 wh/\alpha T a^2$
$10^2 M_y/E\alpha T h^2$
$10^2 M_x/E\alpha T h^2$
$10^3 M_{xy}/E\alpha T h^2$
ALL EDGES SIMPLY SUPPORTED
$a^2/hR_1=10$
$b/a=1$
$R_1/R_2=0.5$
$a^2/hR_1=10$
$b/a=1.2$
$R_1/R_2=0.5$
$10^3 wh/\alpha T a^2$
$10^3 M_{xy}/E\alpha T h^2$
$10^2 M_y/E\alpha T h^2$
$10^2 M_x/E\alpha T h^2$
y/b

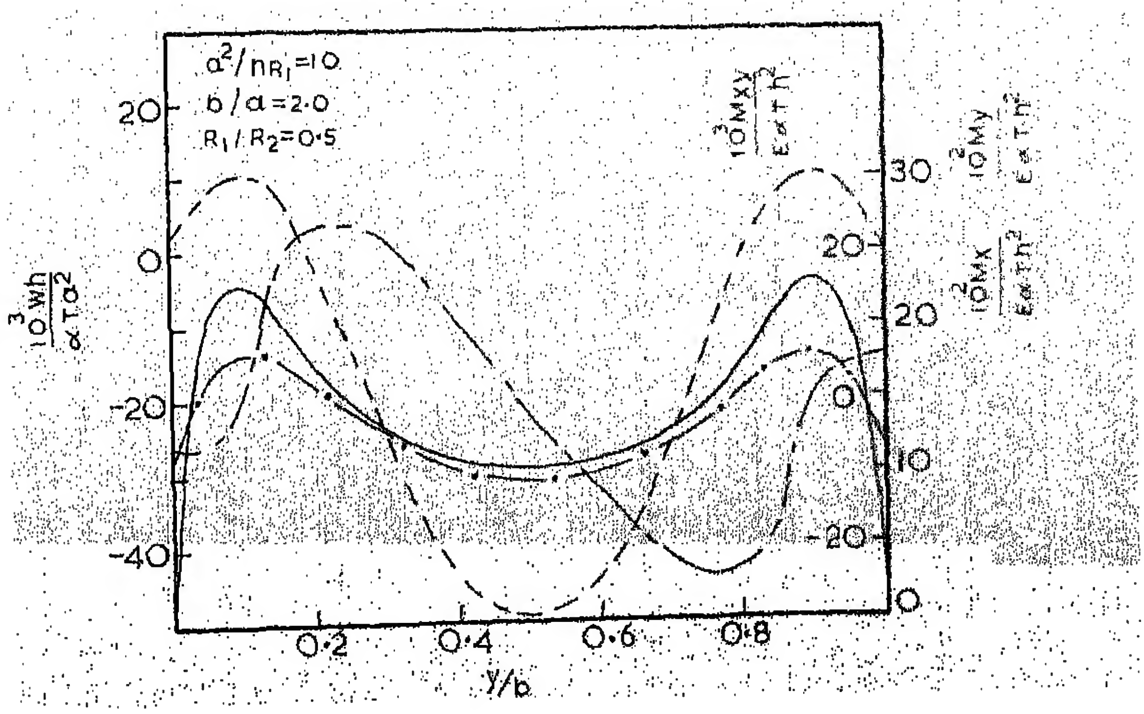
$10^3 wh/\alpha T a^2$ - - -
$10^2 M_y/E\alpha T h^2$ ———
$10^2 M_x/E\alpha T h^2$ —·—·—
$10^3 M_{xy}/E\alpha T h^2$ — — —

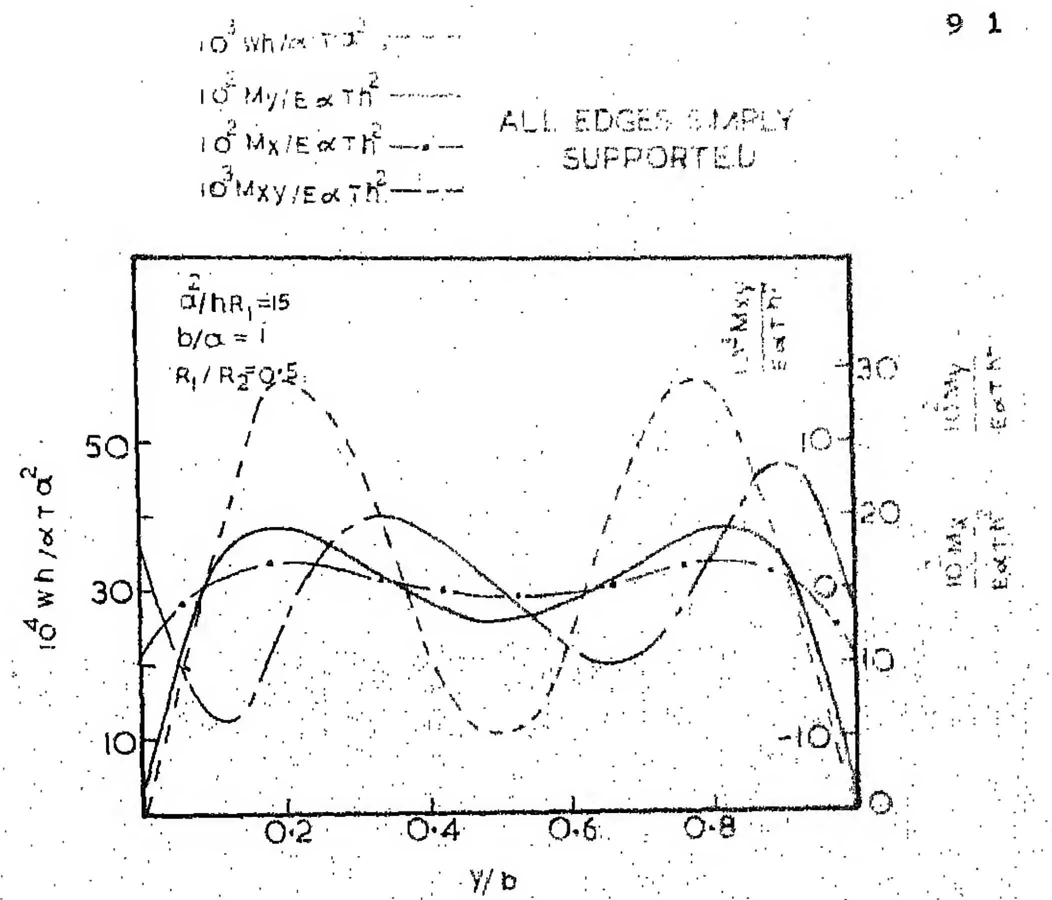
ALL EDGES SIMPLY SUPPORTED

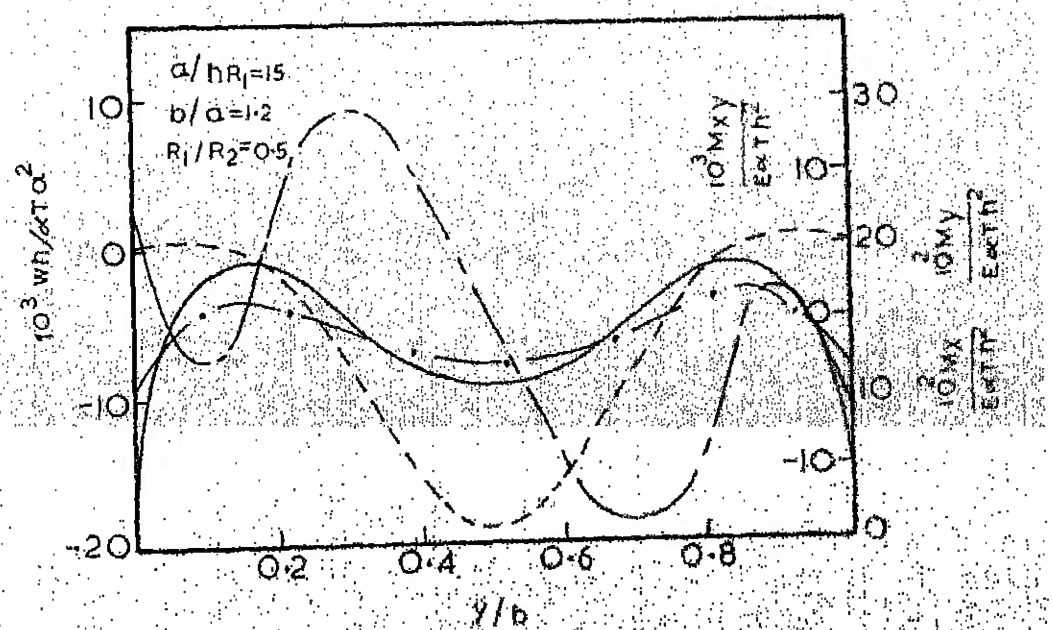


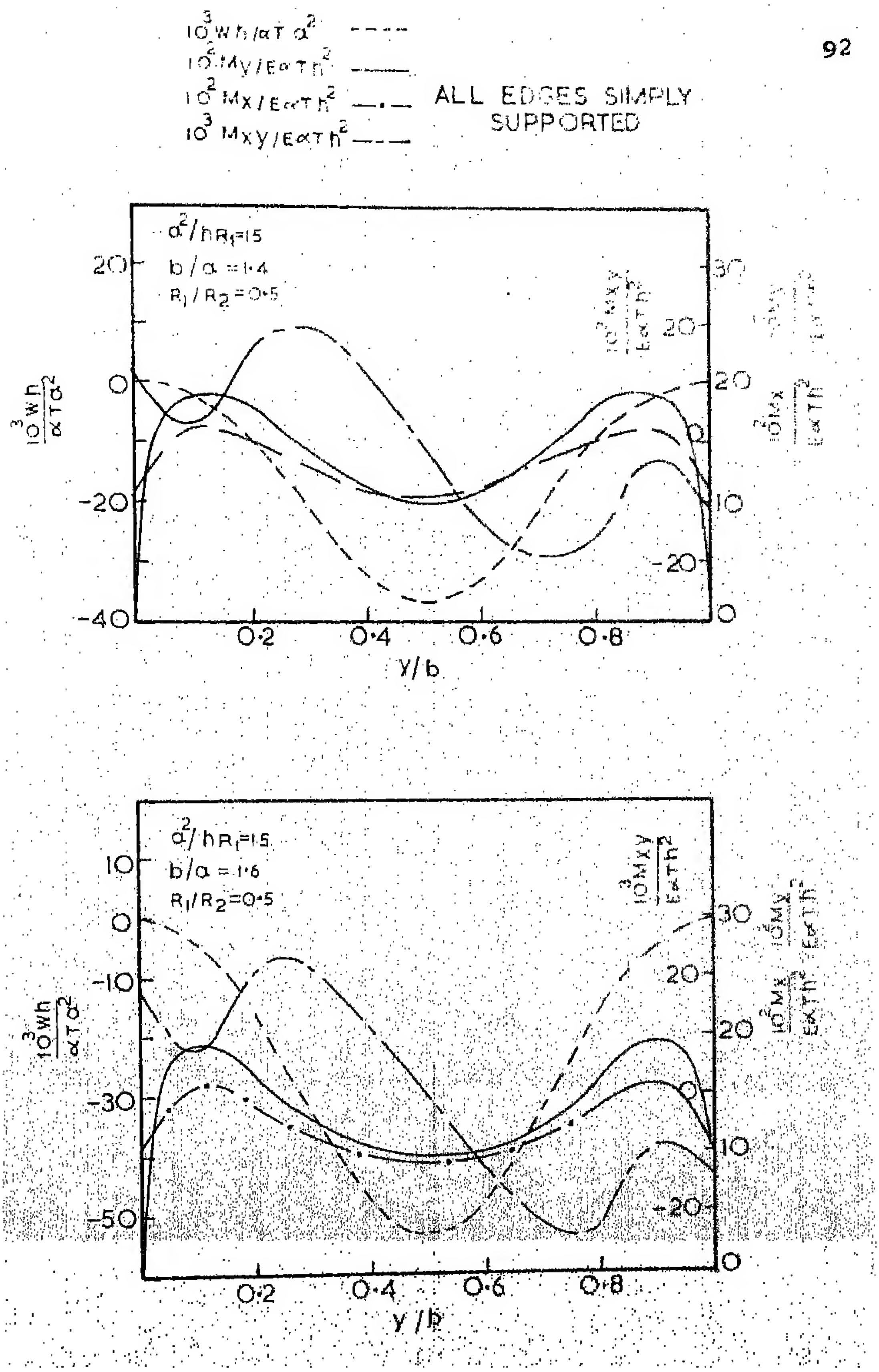
ALL EDGES SIMPLY SUPPORTED



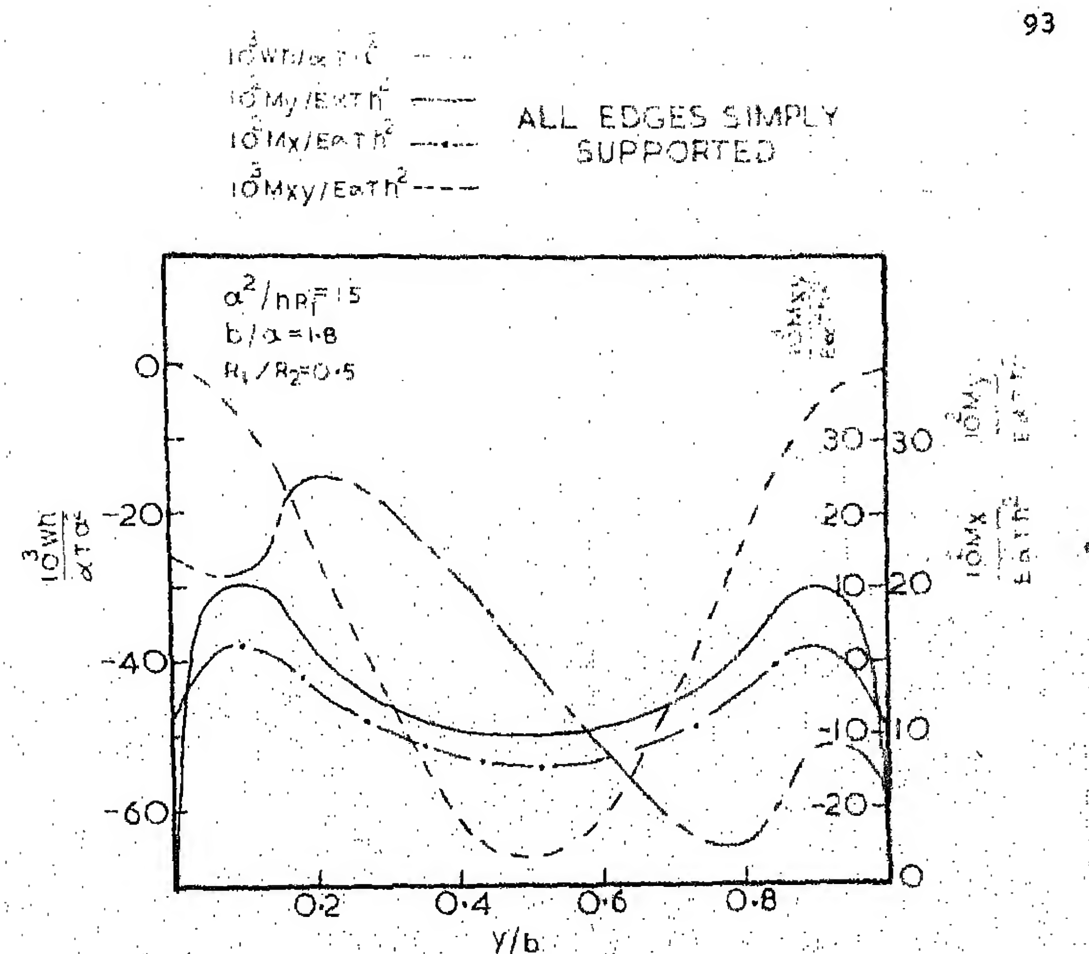


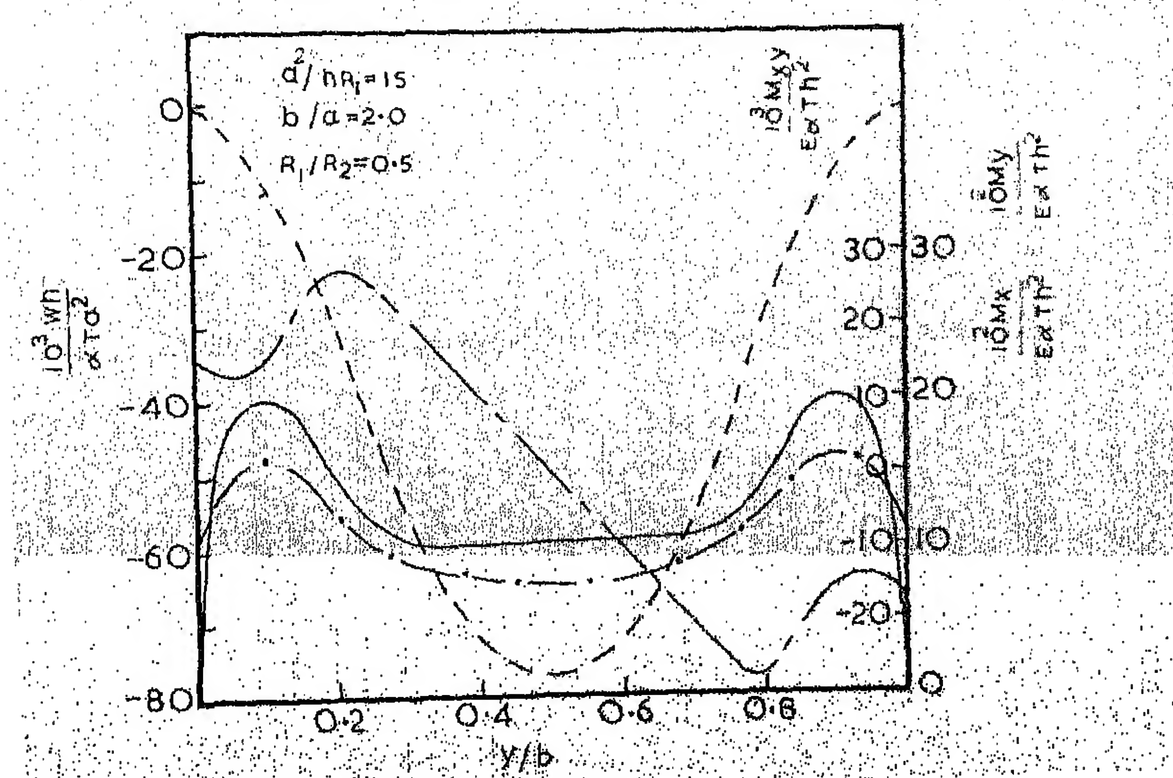

$10^3\, wh/\alpha T a^2$
$10^2\, My/E\alpha T h^2$
$10^2\, Mx/E\alpha T h^2$
$10^3\, Mxy/E\alpha T h^2$
ALL EDGES SIMPLY SUPPORTED
$a^2/hR_1 = 15$
$b/a = 1$
$R_1/R_2 = 0.5$
$10^4\, wh/\alpha T a^2$
y/b

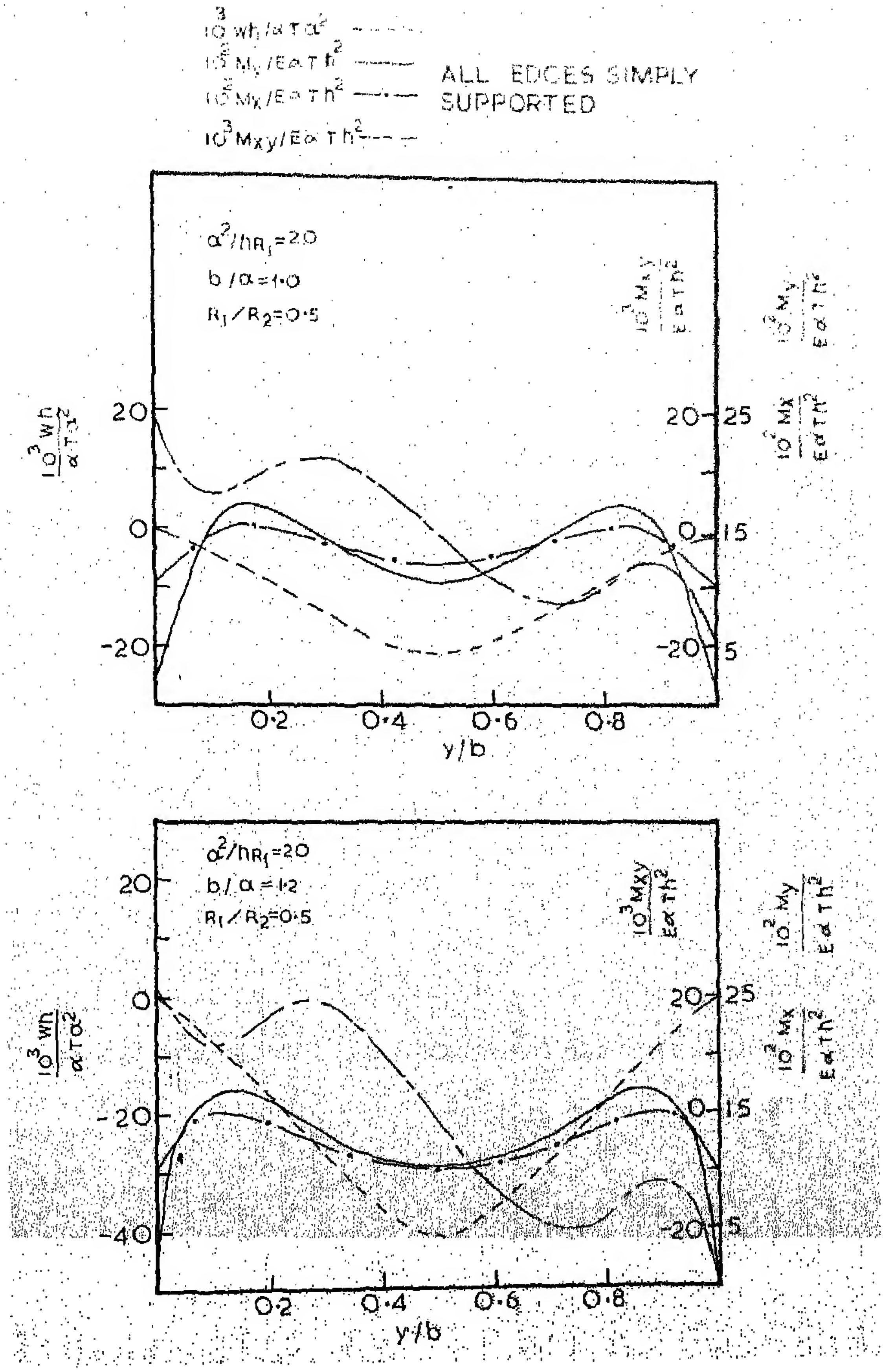

$a^2/hR_1 = 15$
$b/a = 1.2$
$R_1/R_2 = 0.5$
$10^3\, wh/\alpha T a^2$
y/b



$10^3 wh/\alpha T a^2$
$10^2 My/E\alpha Th^2$
$10^2 Mx/E\alpha Th^2$
$10^3 Mxy/E\alpha Th^2$
ALL EDGES SIMPLY SUPPORTED
$a^2/hR_1=15$
$b/a=1{\cdot}4$
$R_1/R_2=0{\cdot}5$
$\frac{10^3 wh}{\alpha T a^2}$
$\frac{10^2 Mx}{E\alpha Th^2}$
y/b
$a^2/hR_1=15$
$b/a=1{\cdot}6$
$R_1/R_2=0{\cdot}5$
$\frac{10^3 Mxy}{E\alpha Th^2}$
$\frac{10^2 My}{E\alpha Th^2}$
$\frac{10^2 Mx}{E\alpha Th^2}$
y/b


ALL EDGES SIMPLY SUPPORTED
$a^2/hR_1 = 15$
$b/a = 1.8$
$R_1/R_2 = 0.5$
y/b

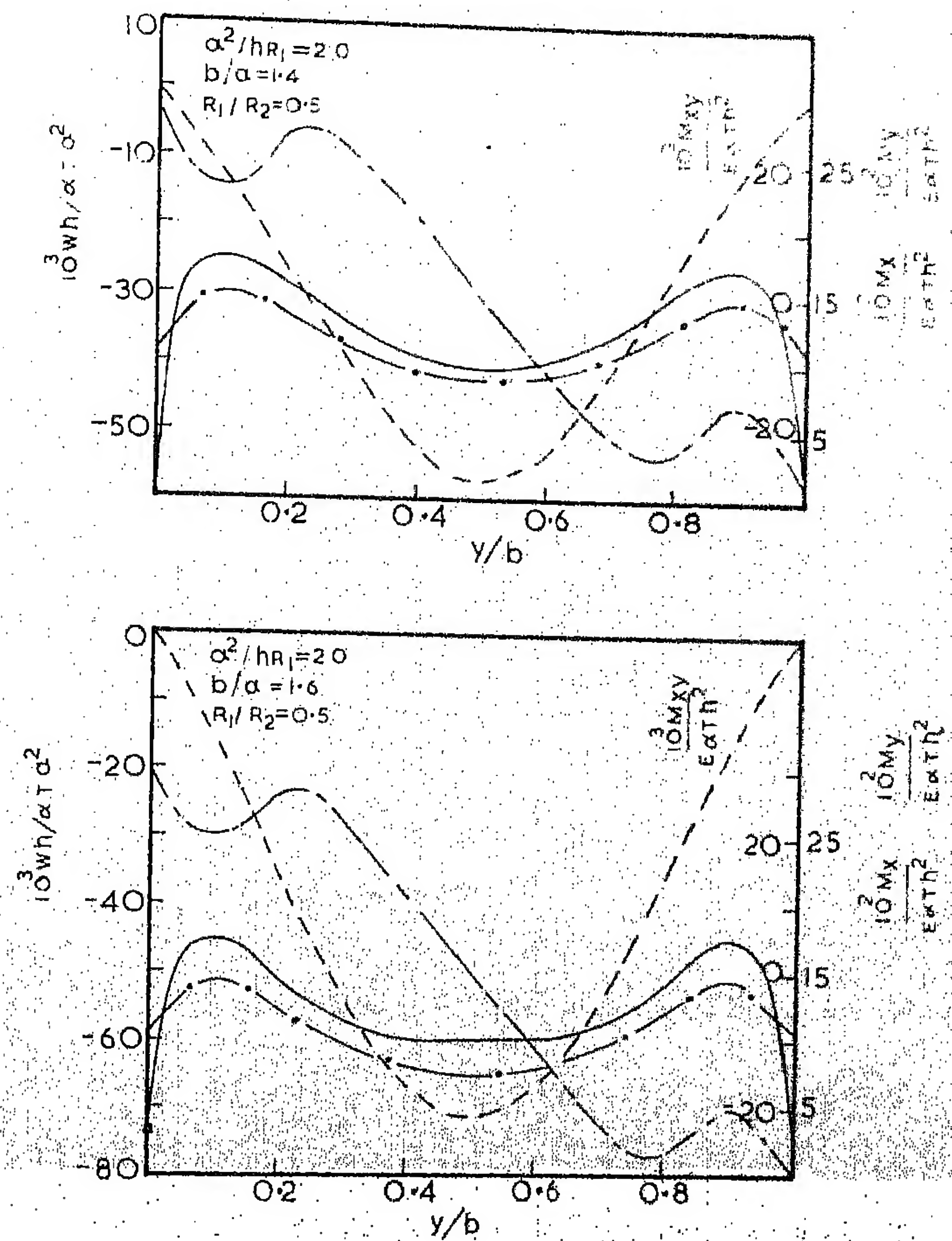

$a^2/hR_1 = 15$
$b/a = 2.0$
$R_1/R_2 = 0.5$
y/b

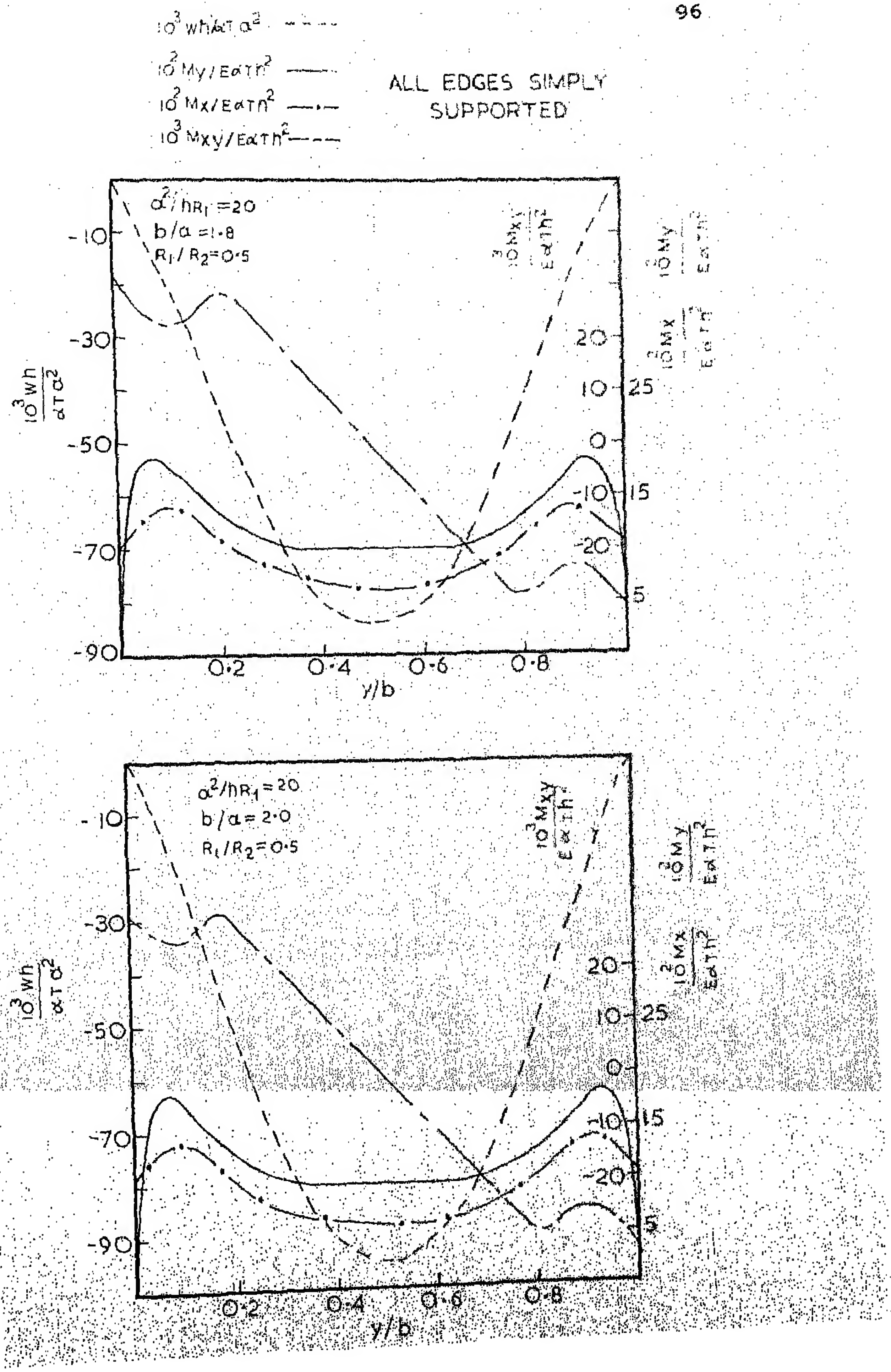


$10^3 wh/\alpha T a^2$ - - - -
$10^2 M_y/E\alpha T h^2$ ———
$10^2 M_x/E\alpha T h^2$ —·—
$10^3 M_{xy}/E\alpha T h^2$ — — —
ALL EDGES SIMPLY SUPPORTED
$a^2/hR_1=20$
$b/a=1{\cdot}0$
$R_1/R_2=0{\cdot}5$
$\frac{10^3 wh}{\alpha T a^2}$
$\frac{10^3 M_{xy}}{E\alpha T h^2}$
$\frac{10^2 M_y}{E\alpha T h^2}$
$\frac{10^2 M_x}{E\alpha T h^2}$
y/b
$a^2/hR_1=20$
$b/a=1{\cdot}2$
$R_1/R_2=0{\cdot}5$
y/b

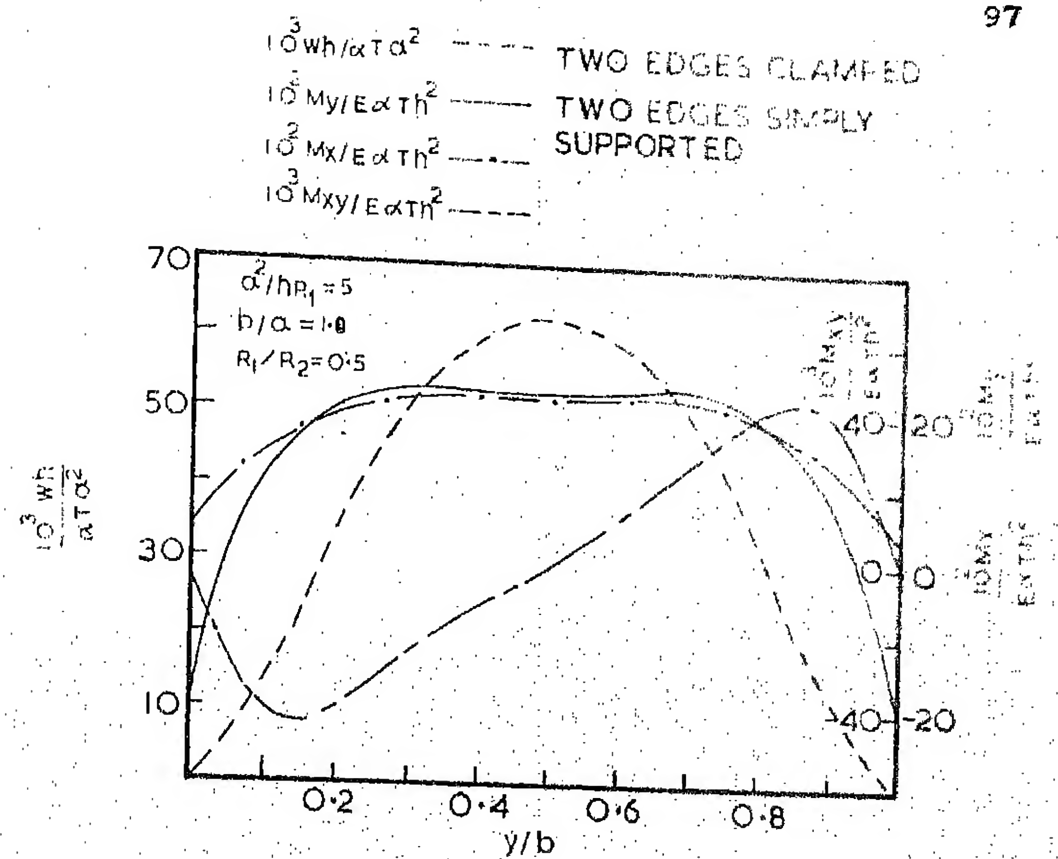
$10^3\ wh/\alpha T a^2$ ---
$10^2\ My/E\alpha T h^2$ ——
$10^2 Mx/E\alpha T h^2$ —·—
$10^3 Mxy/E\alpha T h^2$ —-—

ALL EDGES SIMPLY SUPPORTED

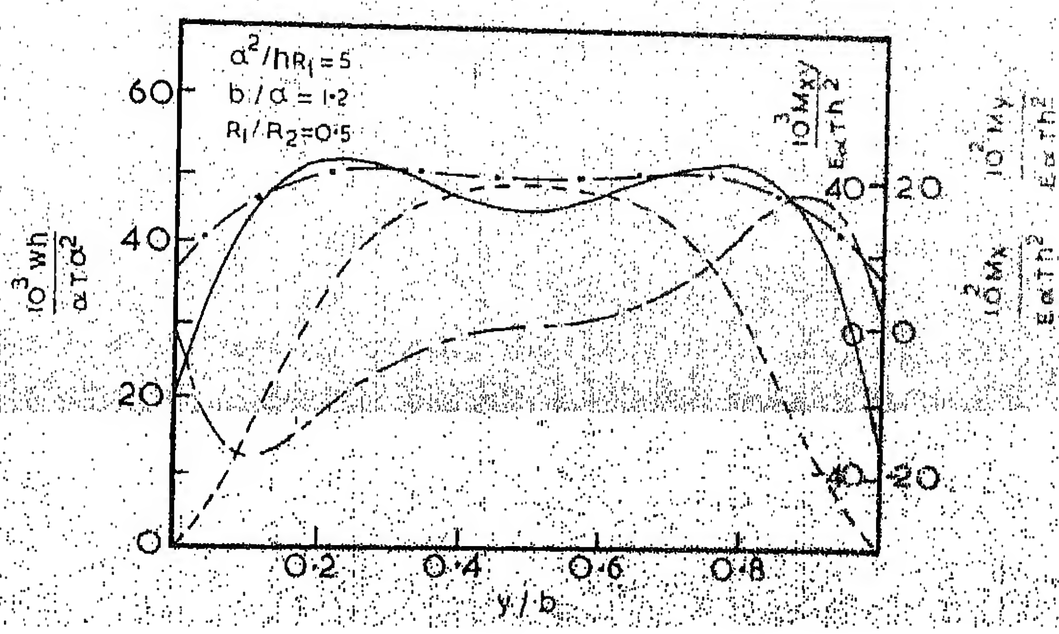




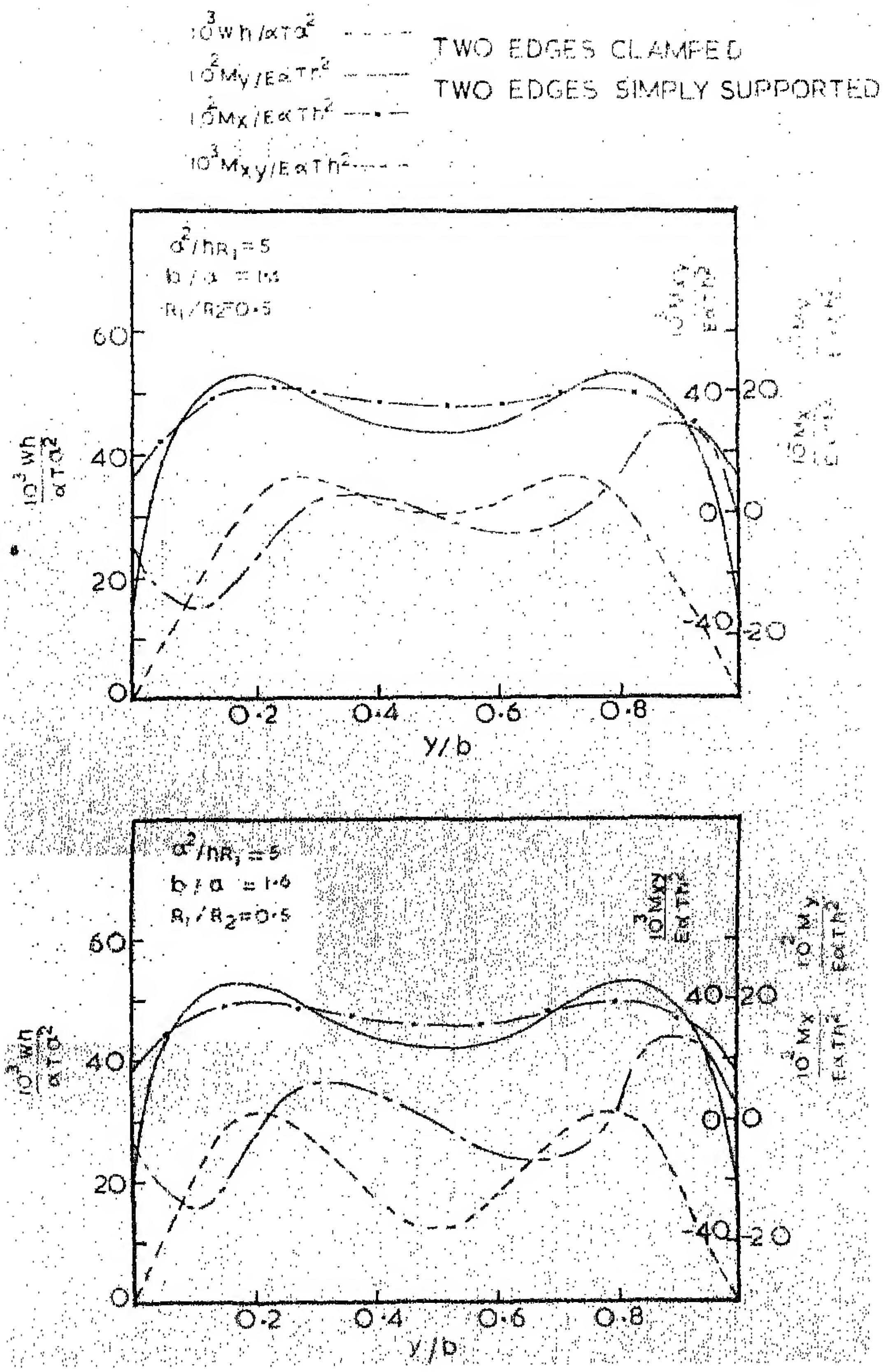
10³ wh/αT a² - - - -
10² My/EαTh² ———
10² Mx/EαTh² —·—
10³ Mxy/EαTh² — — —
ALL EDGES SIMPLY SUPPORTED
a²/hR₁ = 20
b/a = 1·8
R₁/R₂ = 0·5
10³wh/αTa²
10³Mxy/Eαth²
10²My/Eαth²
10²Mx/Eαth²
-10
-30
-50
-70
-90
20
10
0
-10
-20
25
15
5
0·2
0·4
0·6
0·8
y/b
a²/hR₁ = 20
b/a = 2·0
R₁/R₂ = 0·5
10³wh/αTa²
10³Mxy/Eαth²
10²My/Eαth²
10²Mx/Eαth²
-10
-30
-50
-70
-90
20
10
0
-10
-20
25
15
5
0·2
0·4
0·6
0·8
y/b



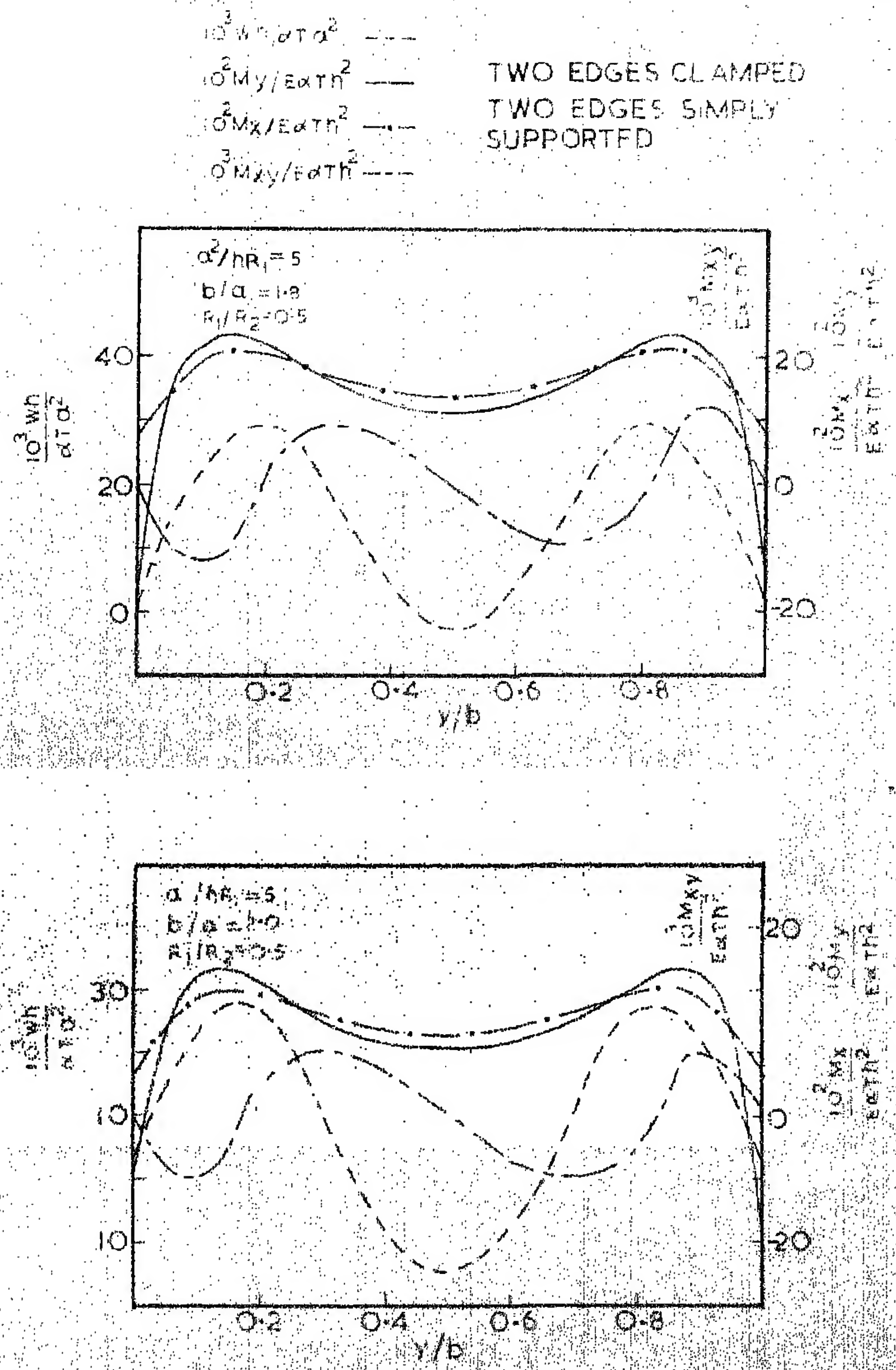
$10^3 wh/\alpha T a^2$
$10^2 My/E\alpha Th^2$
$10^2 Mx/E\alpha Th^2$
$10^3 Mxy/E\alpha Th^2$
TWO EDGES CLAMPED
TWO EDGES SIMPLY SUPPORTED
$a^2/hR_1 = 5$
$b/a = 1.0$
$R_1/R_2 = 0.5$
$\frac{10^3 wh}{\alpha T a^2}$
$\frac{10^3 Mxy}{E\alpha Th^2}$
$\frac{10^2 My}{E\alpha Th^2}$
$\frac{10^2 Mx}{E\alpha Th^2}$
y/b



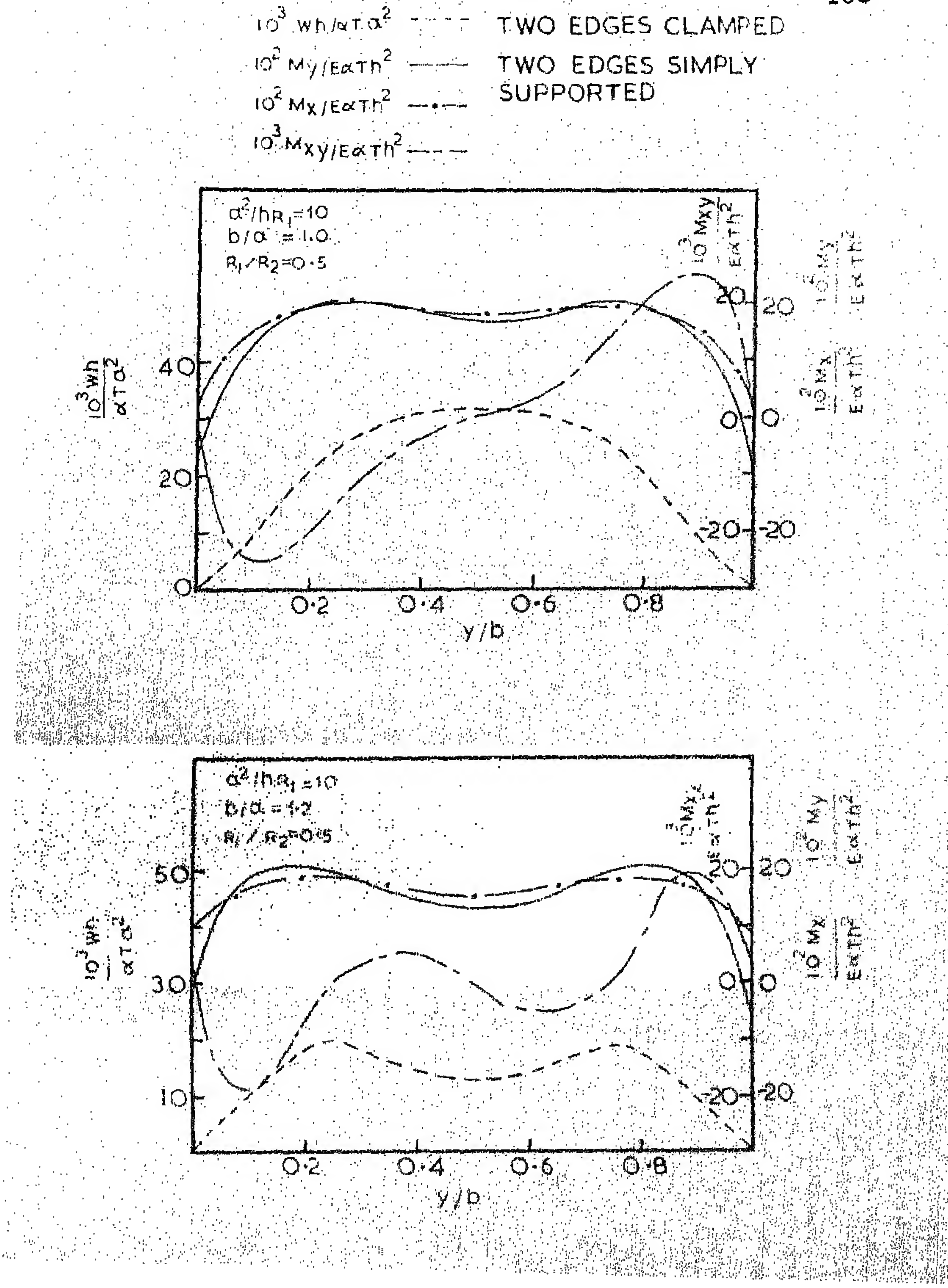
$a^2/hR_1 = 5$
$b/a = 1.2$
$R_1/R_2 = 0.5$
$\frac{10^3 wh}{\alpha T a^2}$
$\frac{10^3 Mxy}{E\alpha Th^2}$
$\frac{10^2 My}{E\alpha Th^2}$
$\frac{10^2 Mx}{E\alpha Th^2}$
y/b



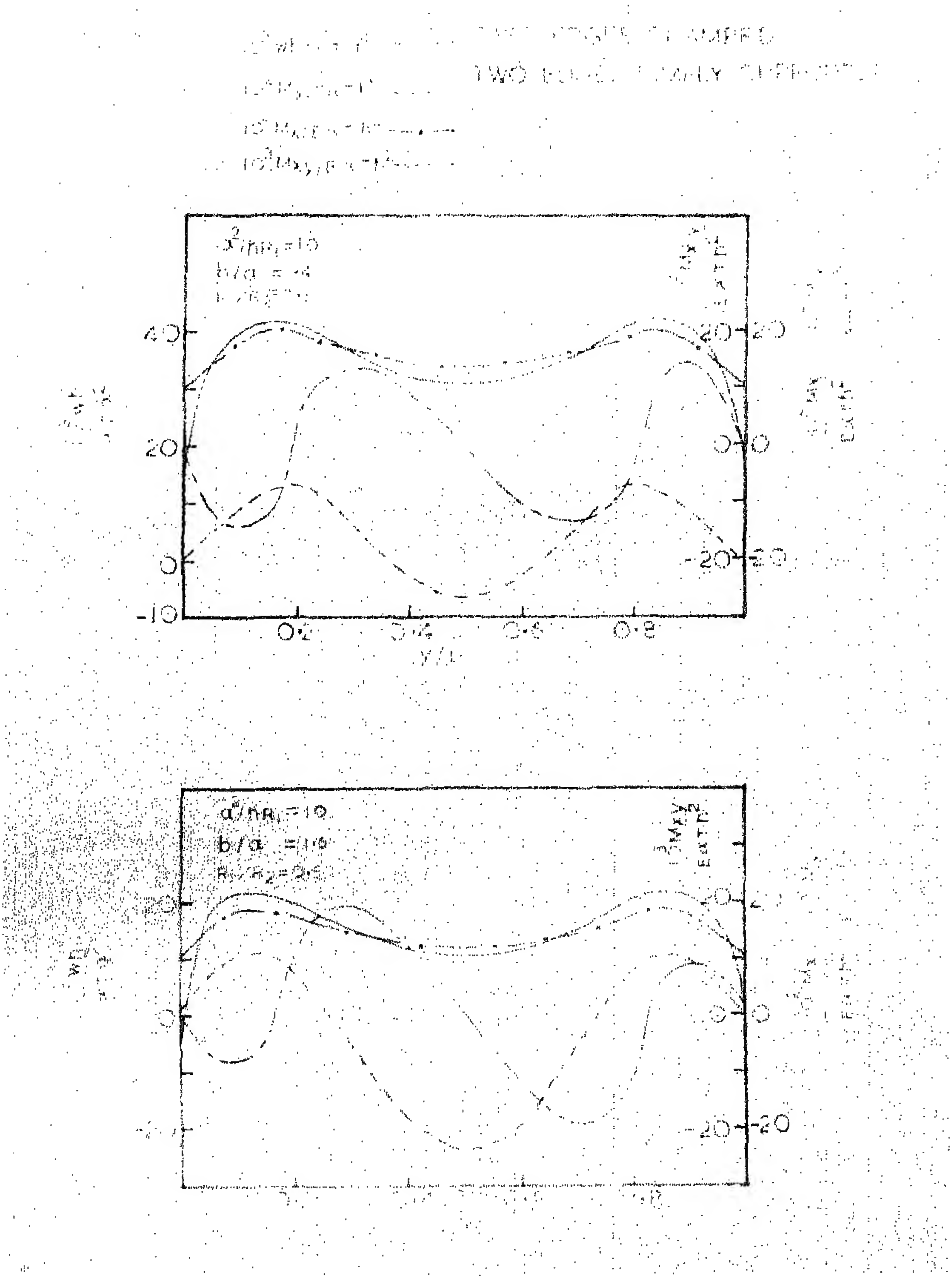
$10^3 wh/\alpha T a^2$
$10^2 M_y/E\alpha T h^2$
$10^2 M_x/E\alpha T h^2$
$10^3 M_{xy}/E\alpha T h^2$
TWO EDGES CLAMPED
TWO EDGES SIMPLY SUPPORTED
$a^2/hR_1=5$
$b/a=1.4$
$R_1/R_2=0.5$
$a^2/hR_1=5$
$b/a=1.6$
$R_1/R_2=0.5$
$\frac{10^3 wh}{\alpha T a^2}$
$\frac{10^3 M_{xy}}{E\alpha T h^2}$
$\frac{10^2 M_y}{E\alpha T h^2}$
$\frac{10^2 M_x}{E\alpha T h^2}$
y/b

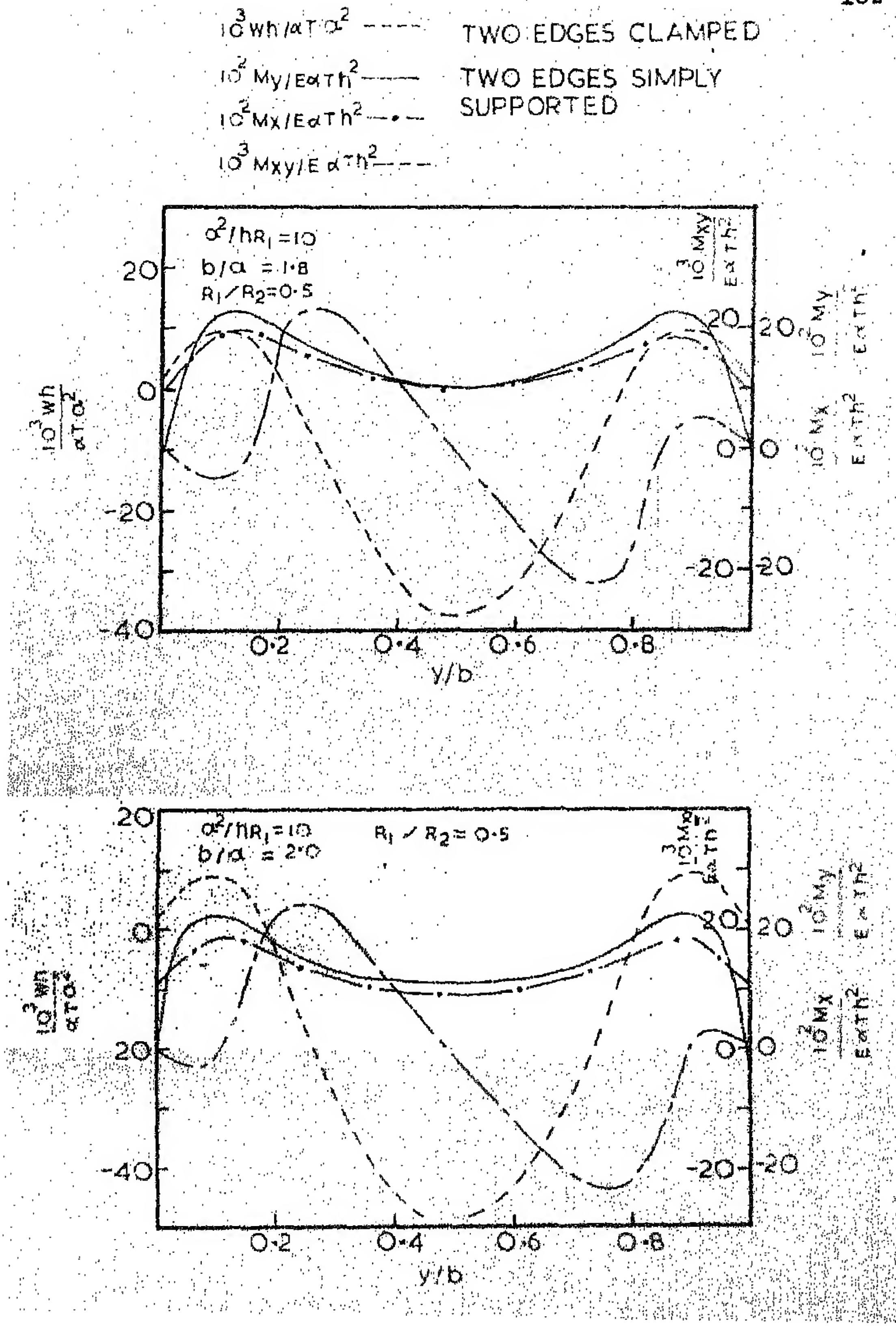


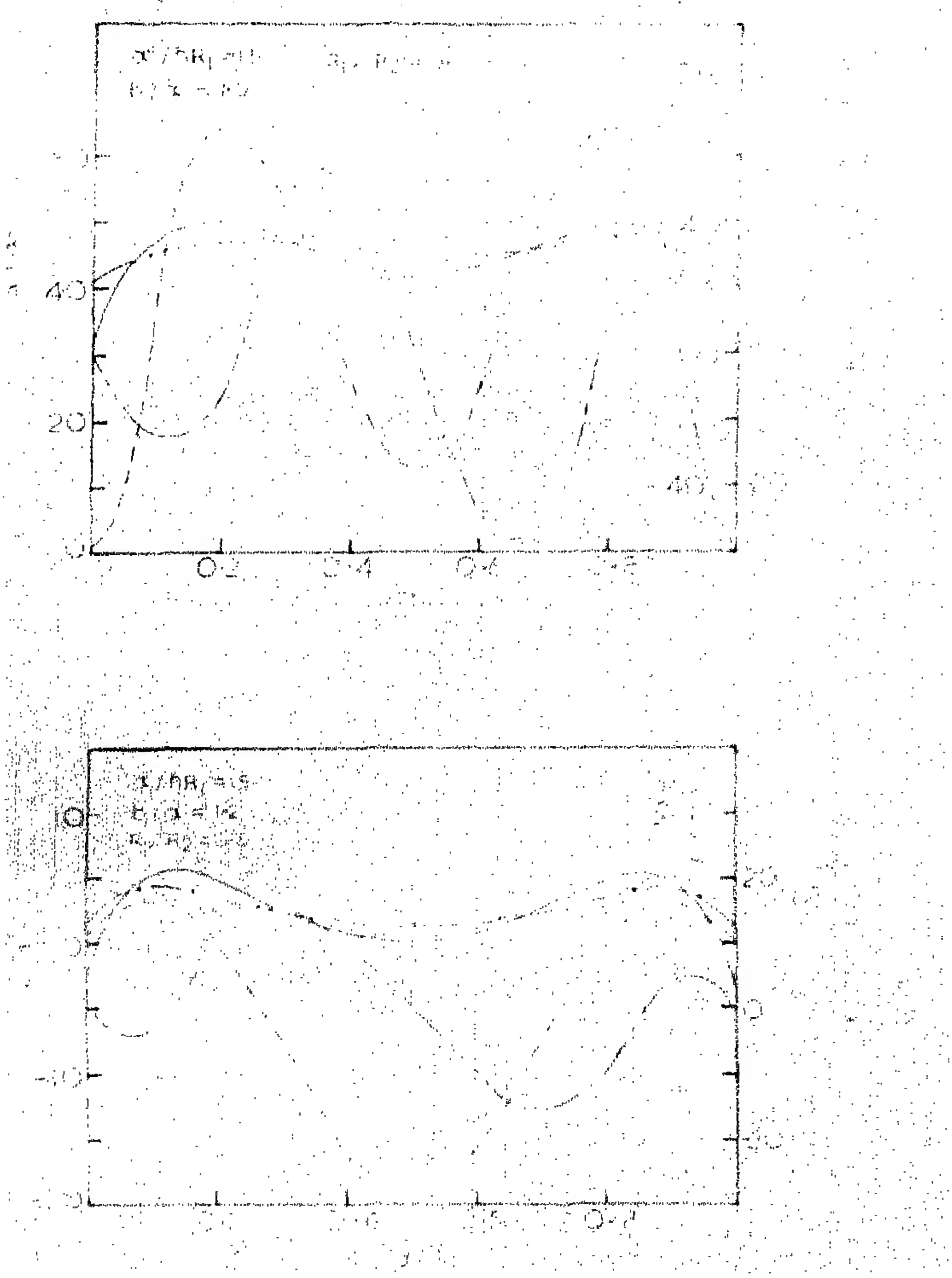
$10^3 wh/\alpha T a^2$
$10^2 M_y/E\alpha Th^2$
$10^2 M_x/E\alpha Th^2$
$10^3 M_{xy}/E\alpha Th^2$
TWO EDGES CLAMPED
TWO EDGES SIMPLY SUPPORTED
$a^2/hR_1 = 5$
$b/a = 1.8$
$R_1/R_2 = 0.5$
$\frac{10^3 wh}{\alpha T a^2}$
$\frac{10^3 M_{xy}}{E\alpha Th^2}$
$\frac{10^2 M_x}{E\alpha Th^2}$
40
20
0
20
0
-20
0·2
0·4
0·6
0·8
y/b
$a^2/hR_1 = 5$
$b/a = 1.0$
$R_1/R_2 = 0.5$
$\frac{10^3 wh}{\alpha T a^2}$
$\frac{10^3 M_{xy}}{E\alpha Th^2}$
$\frac{10^2 M_y}{E\alpha Th^2}$
$\frac{10^2 M_x}{E\alpha Th^2}$
30
10
10
20
0
-20
0·2
0·4
0·6
0·8
y/b

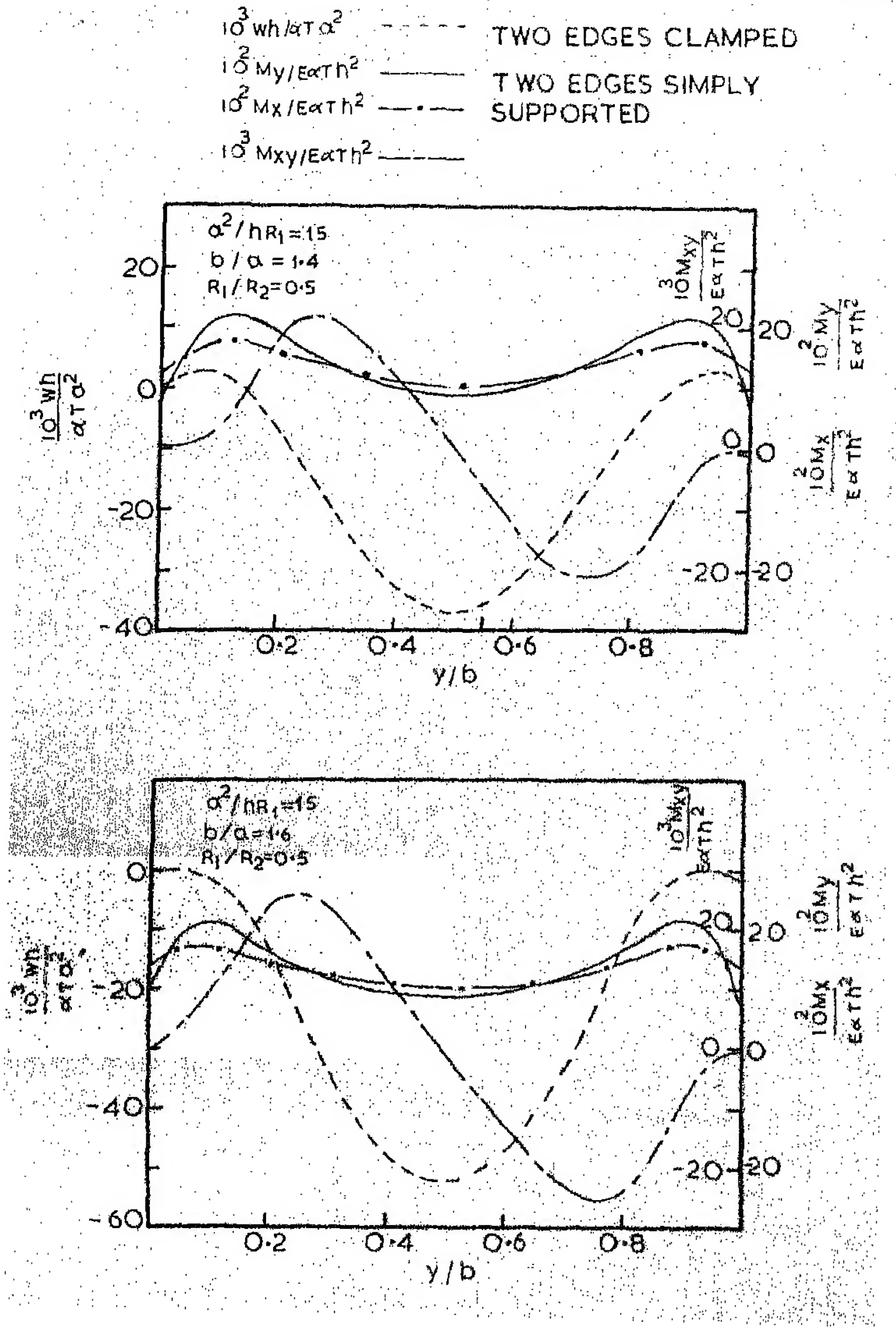


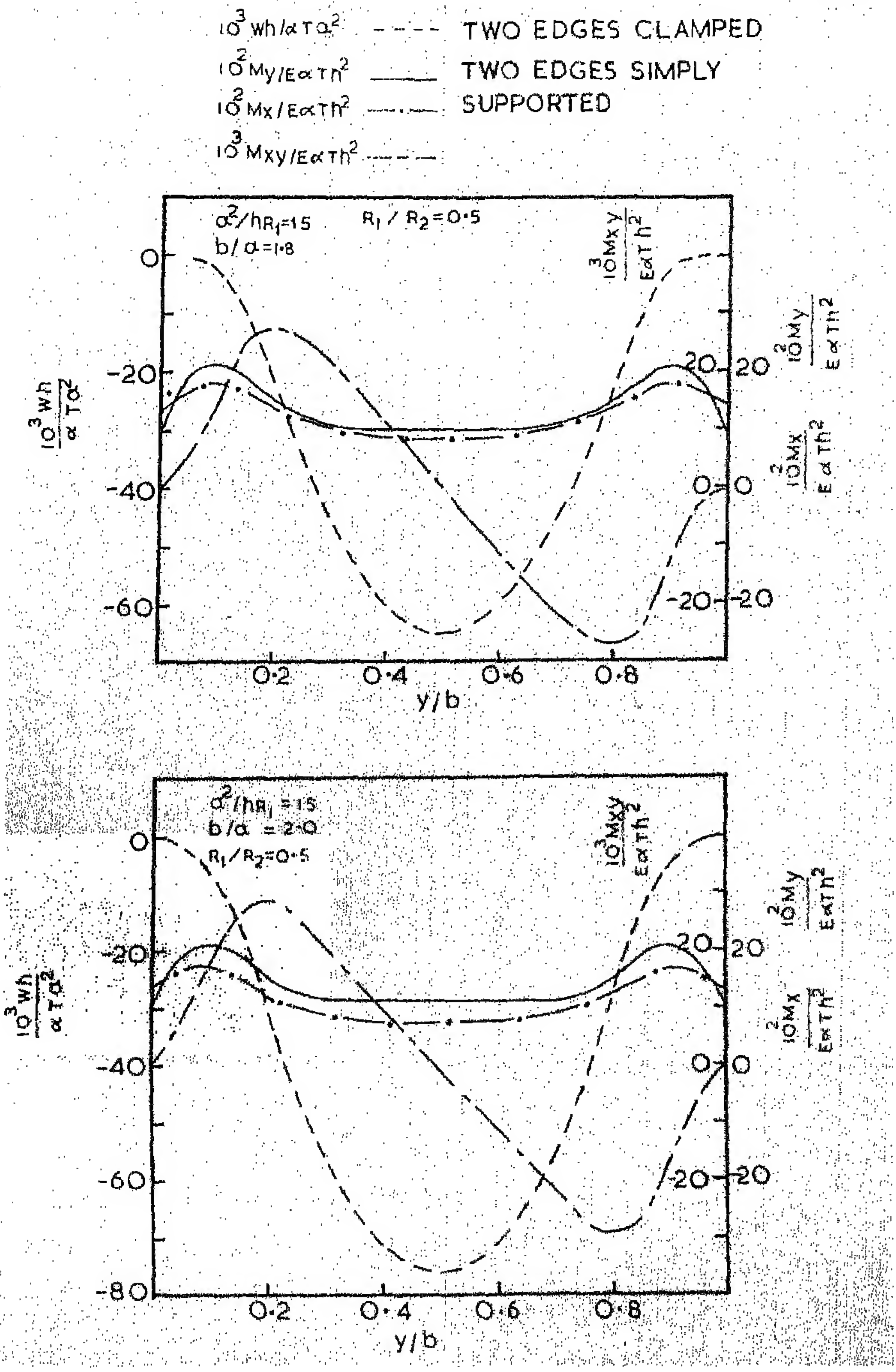
$10^3\ wh/\alpha T a^2$ — TWO EDGES CLAMPED
$10^2\ My/E\alpha Th^2$ — TWO EDGES SIMPLY SUPPORTED
$10^2\ Mx/E\alpha Th^2$
$10^3\ Mxy/E\alpha Th^2$
$a^2/hR_1 = 10$
$b/a = 1.0$
$R_1/R_2 = 0.5$
$a^2/hR_1 = 10$
$b/a = 1.2$
$R_1/R_2 = 0.5$
y/b

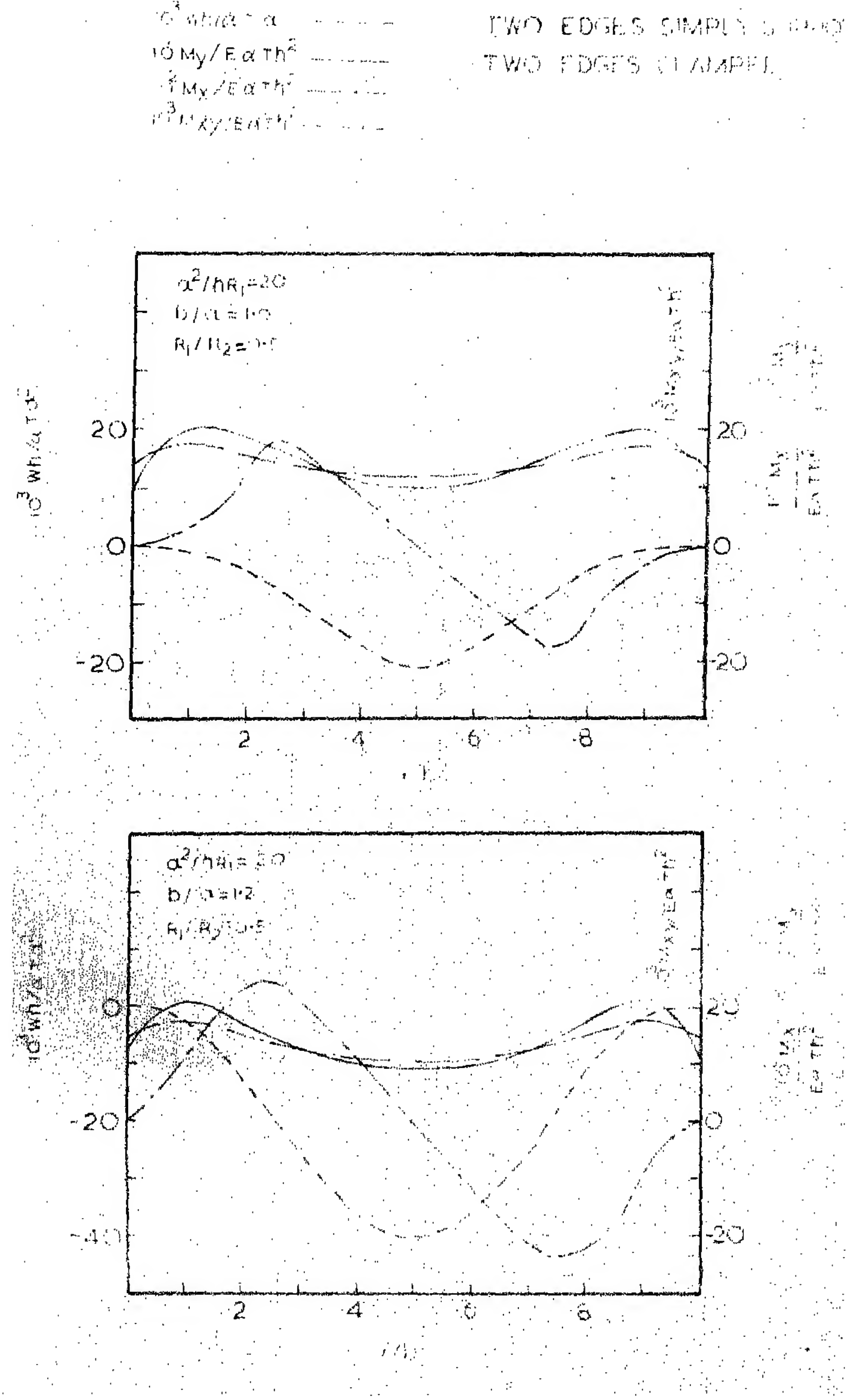



$10^3 wh/\alpha T a^2$ ----
$10^2 My/E\alpha Th^2$ ——
$10^2 Mx/E\alpha Th^2$ —·—
$10^3 Mxy/E\alpha Th^2$ — — —
TWO EDGES CLAMPED
TWO EDGES SIMPLY SUPPORTED
$a^2/hR_1 = 10$
$b/a = 1{\cdot}8$
$R_1/R_2 = 0{\cdot}5$
$\frac{10^3 wh}{\alpha T a^2}$
$\frac{10^3 Mxy}{E\alpha Th^2}$
$\frac{10^2 My}{E\alpha Th^2}$
$\frac{10^2 Mx}{E\alpha Th^2}$
y/b
$a^2/hR_1 = 10$
$b/a = 2{\cdot}0$
$R_1/R_2 = 0{\cdot}5$
y/b

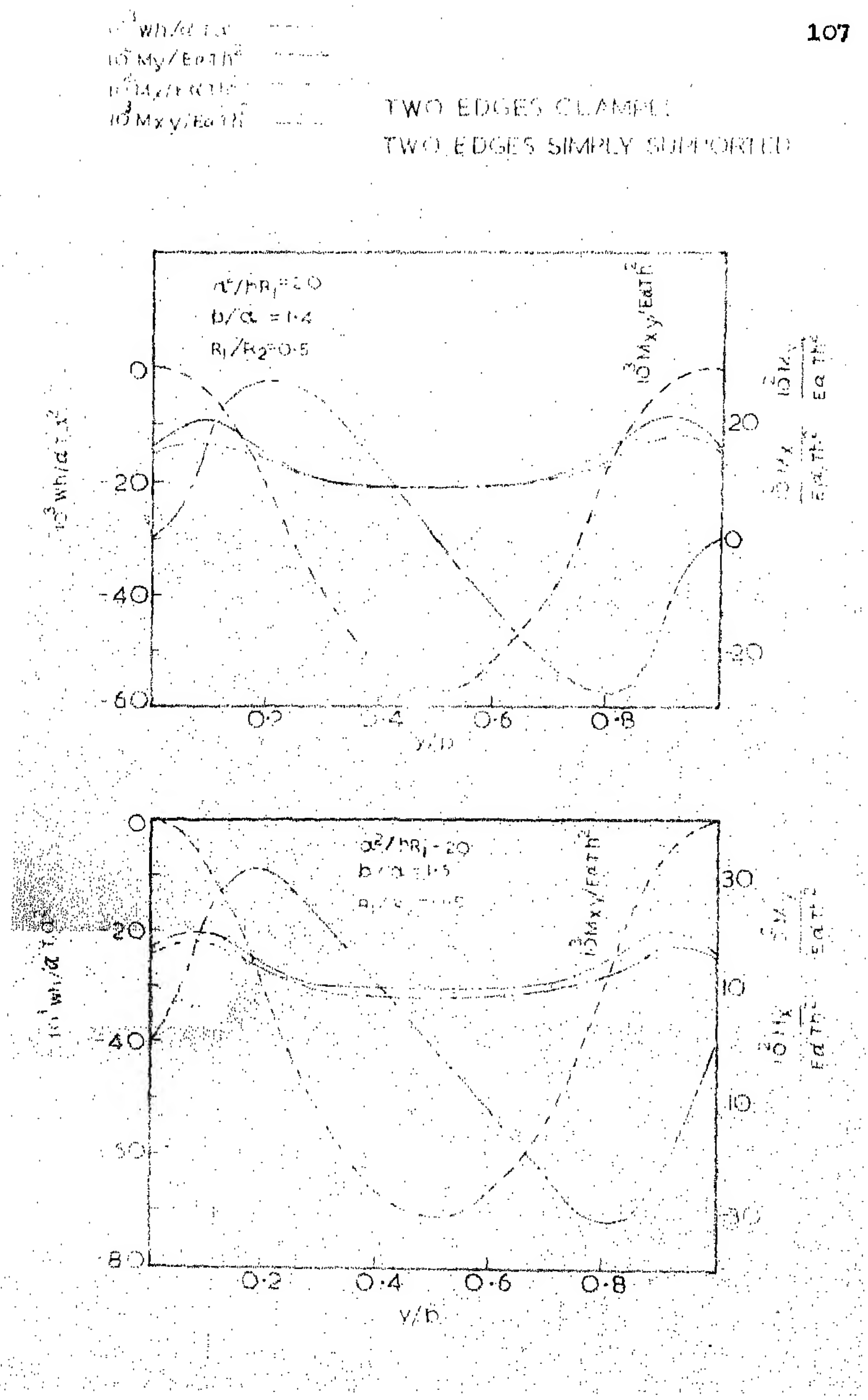


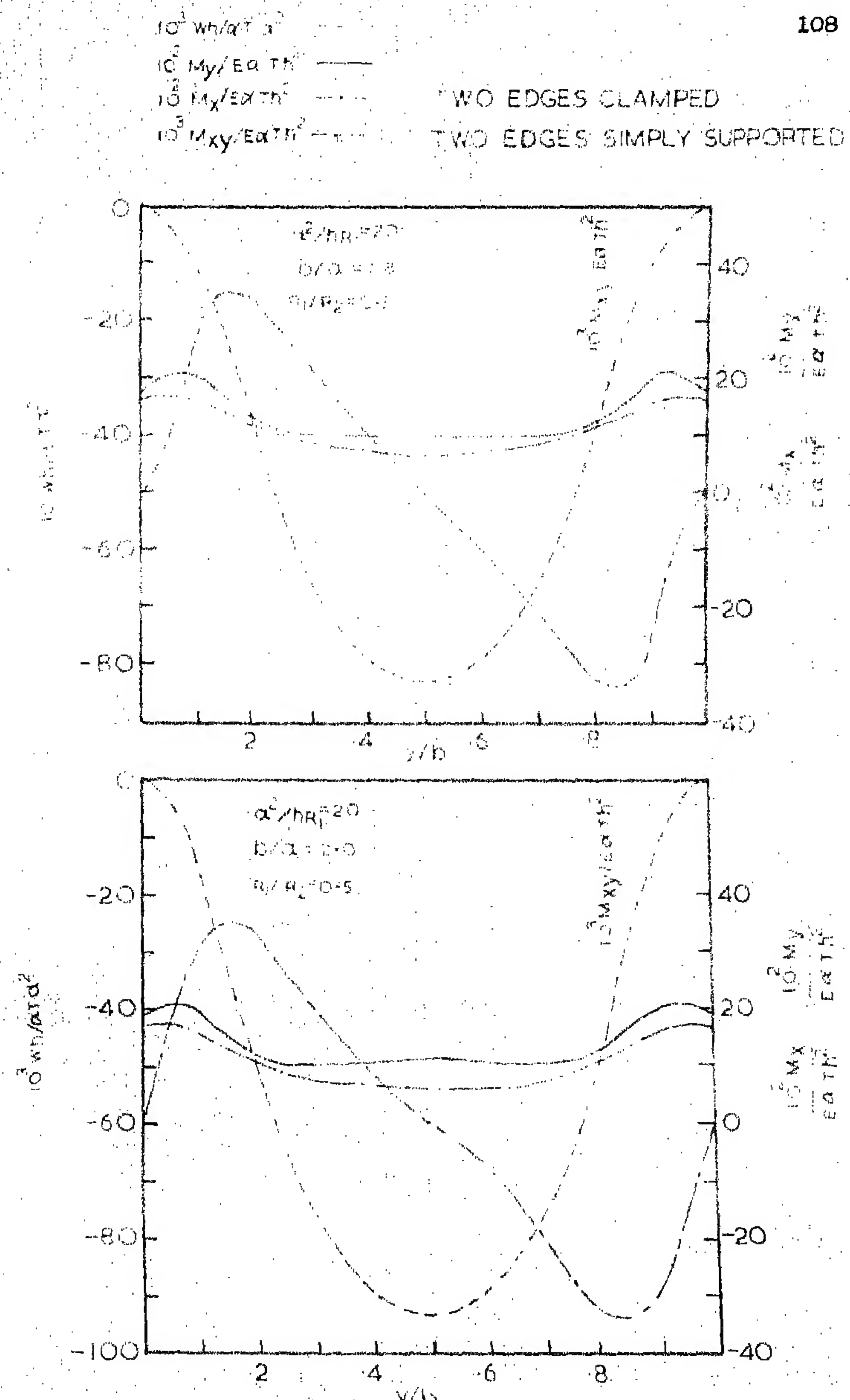

$10^3 wh/\alpha T a^2$
$10^2 M_y/E\alpha T h^2$
$10^2 M_x/E\alpha T h^2$
$10^3 M_{xy}/E\alpha T h^2$
TWO EDGES CLAMPED
TWO EDGES SIMPLY SUPPORTED
$a^2/hR_1=15$
$b/a=1.4$
$R_1/R_2=0.5$
$a^2/hR_1=15$
$b/a=1.6$
$R_1/R_2=0.5$
$\frac{10^3 wh}{\alpha T a^2}$
$\frac{10^3 M_{xy}}{E\alpha T h^2}$
$\frac{10^2 M_y}{E\alpha T h^2}$
$\frac{10^2 M_x}{E\alpha T h^2}$
y/b
0.2
0.4
0.6
0.8


$10^3 wh/\alpha T a^2$
$10^2 M_y/E\alpha Th^2$
$10^2 M_x/E\alpha Th^2$
$10^3 M_{xy}/E\alpha Th^2$
TWO EDGES CLAMPED
TWO EDGES SIMPLY SUPPORTED
$a^2/hR_1=15$
$b/a=1.8$
$R_1/R_2=0.5$
$a^2/hR_1=15$
$b/a=2.0$
$R_1/R_2=0.5$
y/b


TWO EDGES SIMPLY
TWO EDGES CLAMPED
20
0
-20
2
4
6
8
-20
-40
20
0
-20
2
4
6
6

TWO EDGES CLAMPED
TWO EDGES SIMPLY SUPPORTED





TWO EDGES CLAMPED
TWO EDGES SIMPLY SUPPORTED
b/a = 2·0
R1/R2 = 0·5
y/b

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